Problem Set # 11 Diff. Eqns Handout 6, 7, 8, 9, 10, 11

(6) a) $\frac{dM}{dt} = .04M \quad M(0) = 2000$

b) $\int \frac{dM}{M} = \int .04\,dt \Rightarrow M = Ce^{.04t}$

$M(0) = C = 2000$ | $M(t) = 2000\,e^{.04t}$

c) $\frac{dM}{dt} = .04M + 1000 \quad M(0) = 2000$

d) $\int \frac{dM}{.04M + 1000} = \int dt \Rightarrow \ln|.04M + 1000| = .04t + C_1$

$.04M + 1000 = C_2 e^{.04t}$

$M(t) = \frac{C_2 e^{.04t} - 1000}{.04}$

$M(0) = \frac{C_1 - 1000}{.04} = 2000$

$C_1 = 1080$

$M(t) = \frac{1080\,e^{.04t} - 1000}{.04}$

(7) $\frac{dB}{dt} = .0725B - 12000$

$B(t) = Ce^{\frac{29}{400}t} + 165,517$

$B(0) = C + 165,517 = 100,000$

$C = -65,517$

$B(t) = -65,517\,e^{\frac{29}{400}t} + 165,517$

(8) a) $\frac{dM}{dt} = .05M \quad B(0) = 600$

$B(10) = 800$

b) $\frac{dB}{dt} = kB$

$B = C_1 e^{kt}$

$B(0) = C_1 = 600$

$B(10) = 600e^{10k} = 800$

$k = .02877$

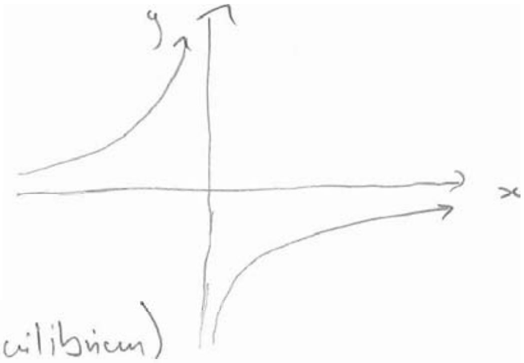
$\frac{dB}{dt} = .02877B$

(9) a) $\frac{dP}{dt} = .03P - 6000$

b) $P(t) = 2{,}800{,}000\, e^{.03t} + 200{,}000$

(10) $\int \frac{dy}{y^2} = \int dx \quad -y^{-1} = x + c \quad \Rightarrow (c+x)y = -1$

$y = \frac{-1}{c+x}$



(a) i. $y(0)=0$ y stays at zero (equilibrium)

ii. Begins at a negative value, then approaches zero

iii. Approaches infinity

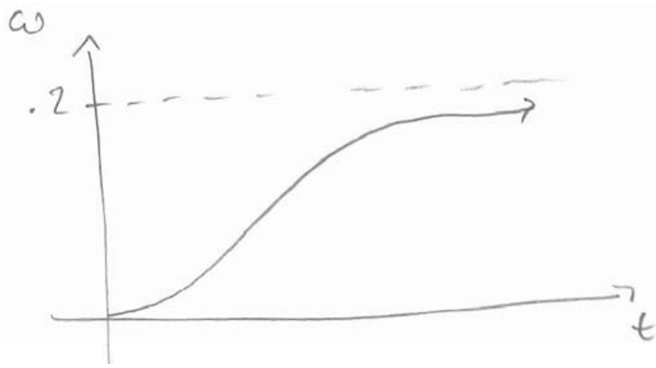
(b) Unstable

(c) $y(0) = \frac{-1}{c} = 1 \Rightarrow c = -1$

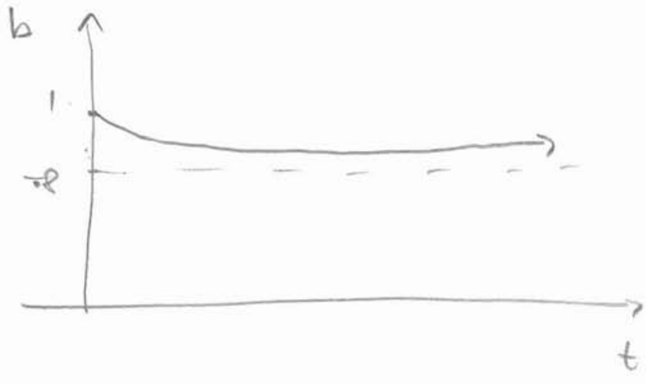
$y(t) = \frac{-1}{-1+x}$

(d) $\lim_{x \to 1} y \to \infty$

(11) a) $\frac{dw}{dt} = .2(2) - 2w \quad w(0) = 0$



b) $\frac{db}{dt} = .8(2) - 2b \quad b(0) = 1$



Problem Set # 11 Supplement 31.3 6, 7, 17, 21

(6) d) Asymptote at $y = -1$

Slope is -ve between $(0, 1)$

(7)

Asymptote at $x = 2$

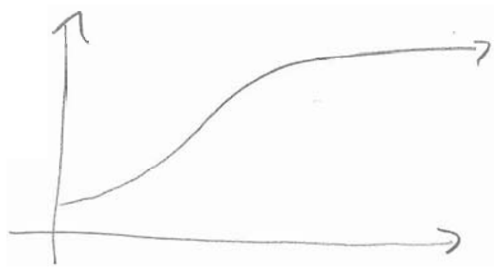
(17) a) $\frac{dy}{dt} = (y+1)(y-4)$

b) $\frac{dy}{dt} = -(y+1)(y-4)$

c) $\frac{dy}{dt} = y(y+1)(y-4)$

(21) 1961 - 1978 = 17 years $\Delta P_1 = 1.25$ mil $\sim .07$ mil/yr

1978 - 1991 = 13 years $\Delta P_2 = .75$ mil $\sim .057$ mil/year



Logistic growth model since pop. growth is slowing

3. $(x^2+1)y' = xy \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{xy}{x^2+1} \;\Rightarrow\; \dfrac{dy}{y} = \dfrac{x\,dx}{x^2+1}\;\; [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int \frac{x\,dx}{x^2+1} \;\Rightarrow$

$\ln|y| = \frac{1}{2}\ln(x^2+1) + C \;\; [u = x^2+1,\, du = 2x\,dx] \; = \ln(x^2+1)^{1/2} + \ln e^C = \ln\left(e^C\sqrt{x^2+1}\right) \;\Rightarrow$

$|y| = e^C\sqrt{x^2+1} \;\Rightarrow\; y = K\sqrt{x^2+1}$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

4. $y' = y^2 \sin x \;\Rightarrow\; \dfrac{dy}{dx} = y^2\sin x \;\Rightarrow\; \dfrac{dy}{y^2} = \sin x\,dx \;\; [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y^2} = \int \sin x\,dx \;\Rightarrow$

$-\dfrac{1}{y} = -\cos x + C \;\Rightarrow\; \dfrac{1}{y} = \cos x - C \;\Rightarrow\; y = \dfrac{1}{\cos x + K}$, where $K = -C$. $y = 0$ is also a solution.

5. $(1+\tan y)\,y' = x^2+1 \;\Rightarrow\; (1+\tan y)\dfrac{dy}{dx} = x^2+1 \;\Rightarrow\; \left(1 + \dfrac{\sin y}{\cos y}\right) dy = (x^2+1)\,dx \;\Rightarrow$

$\displaystyle\int \left(1 - \frac{-\sin y}{\cos y}\right) dy = \int (x^2+1)\,dx \;\Rightarrow\; y - \ln|\cos y| = \tfrac{1}{3}x^3 + x + C$. Note: The left side is equivalent to $y + \ln|\sec y|$.

6. $\dfrac{dy}{d\theta} = \dfrac{e^y\sin^2\theta}{y\sec\theta} \;\Rightarrow\; \dfrac{y}{e^y}\,dy = \dfrac{\sin^2\theta}{\sec\theta}\,d\theta \;\Rightarrow\; \displaystyle\int ye^{-y}\,dy = \int \sin^2\theta\cos\theta\,d\theta$. Integrating the left side by parts with $u = y$, $dv = e^{-y}\,dy$ and the right side by the substitution $u = \sin\theta$, we obtain $-ye^{-y} - e^{-y} = \frac{1}{3}\sin^3\theta + C$. We cannot solve explicitly for y.

7. $\dfrac{du}{dt} = 2 + 2u + t + tu \;\Rightarrow\; \dfrac{du}{dt} = (1+u)(2+t) \;\Rightarrow\; \displaystyle\int \frac{du}{1+u} = \int (2+t)\,dt \;\; [u \neq -1] \;\Rightarrow$

$\ln|1+u| = \frac{1}{2}t^2 + 2t + C \;\Rightarrow\; |1+u| = e^{t^2/2+2t+C} = Ke^{t^2/2+2t}$, where $K = e^C \;\Rightarrow\; 1+u = \pm Ke^{t^2/2+2t} \;\Rightarrow$

$u = -1 \pm Ke^{t^2/2+2t}$ where $K > 0$. $u = -1$ is also a solution, so $u = -1 + Ae^{t^2/2+2t}$, where A is an arbitrary constant.

8. $\dfrac{dz}{dt} + e^{t+z} = 0 \;\Rightarrow\; \dfrac{dz}{dt} = -e^te^z \;\Rightarrow\; \displaystyle\int e^{-z}\,dz = -\int e^t\,dt \;\Rightarrow\; -e^{-z} = -e^t + C \;\Rightarrow\; e^{-z} = e^t - C \;\Rightarrow$

$\dfrac{1}{e^z} = e^t - C \;\Rightarrow\; e^z = \dfrac{1}{e^t - C} \;\Rightarrow\; z = \ln\left(\dfrac{1}{e^t - C}\right) \;\Rightarrow\; z = -\ln(e^t - C)$

9. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}$, $u(0) = -5$. $\int 2u\,du = \int (2t + \sec^2 t)\,dt \;\Rightarrow\; u^2 = t^2 + \tan t + C$, where $[u(0)]^2 = 0^2 + \tan 0 + C \;\Rightarrow\; C = (-5)^2 = 25$. Therefore, $u^2 = t^2 + \tan t + 25$, so $u = \pm\sqrt{t^2 + \tan t + 25}$. Since $u(0) = -5$, we must have $u = -\sqrt{t^2 + \tan t + 25}$.

10. $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2}$, $(0) = 1$. $dy = y\cos x\,dx \;\Rightarrow\; \dfrac{1+y^2}{y}\,dy = \cos x\,dx \;\Rightarrow\; \displaystyle\int \left(\frac{1}{y} + y\right) dy = \int \cos x\,dx \;\Rightarrow$

$\ln|y| + \frac{1}{2}y^2 = \sin x + C$. $y(0) = 1 \;\Rightarrow\; \ln 1 + \frac{1}{2} = \sin 0 + C \;\Rightarrow\; C = \frac{1}{2}$, so $\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}$. We cannot solve explicitly for y.

11. $x\cos x = (2y + e^{3y})\,y' \;\Rightarrow\; x\cos x\,dx = (2y + e^{3y})\,dy \;\Rightarrow\; \int (2y + e^{3y})\,dy = \int x\cos x\,dx \;\Rightarrow$

$y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x + C$ [where the second integral is evaluated using integration by parts]. Now $y(0) = 0 \;\Rightarrow$ $0 + \frac{1}{3} = 0 + 1 + C \;\Rightarrow\; C = -\frac{2}{3}$. Thus, a solution is $y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x - \frac{2}{3}$. We cannot solve explicitly for y.

12. $\dfrac{dP}{dt} = \sqrt{Pt} \;\Rightarrow\; dP/\sqrt{P} = \sqrt{t}\,dt \;\Rightarrow\; \int P^{-1/2}\,dP = \int t^{1/2}\,dt \;\Rightarrow\; 2P^{1/2} = \frac{2}{3}t^{3/2} + C.$

$P(1) = 2 \;\Rightarrow\; 2\sqrt{2} = \frac{2}{3} + C \;\Rightarrow\; C = 2\sqrt{2} - \frac{2}{3}$, so $2P^{1/2} = \frac{2}{3}t^{3/2} + 2\sqrt{2} - \frac{2}{3} \;\Rightarrow\; \sqrt{P} = \frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3} \;\Rightarrow$

$P = \left(\frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3}\right)^2.$

13. $y' \tan x = a + y,\ 0 < x < \pi/2 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{a+y}{\tan x} \;\Rightarrow\; \dfrac{dy}{a+y} = \cot x\,dx \quad [a + y \neq 0] \;\Rightarrow$

$\displaystyle\int \frac{dy}{a+y} = \int \frac{\cos x}{\sin x}\,dx \;\Rightarrow\; \ln|a+y| = \ln|\sin x| + C \;\Rightarrow\; |a+y| = e^{\ln|\sin x| + C} = e^{\ln|\sin x|} \cdot e^C = e^C\,|\sin x| \;\Rightarrow$

$a + y = K \sin x$, where $K = \pm e^C$. (In our derivation, K was nonzero, but we can restore the excluded case $y = -a$ by allowing K to be zero.) $y(\pi/3) = a \;\Rightarrow\; a + a = K\sin(\pi/3) \;\Rightarrow\; 2a = K\dfrac{\sqrt{3}}{2} \;\Rightarrow\; K = \dfrac{4a}{\sqrt{3}}$

Thus, $a + y = \dfrac{4a}{\sqrt{3}}\sin x$ and so $y = \dfrac{4a}{\sqrt{3}}\sin x - a.$

14. $\dfrac{dL}{dt} = kL^2 \ln t \;\Rightarrow\; \dfrac{dL}{L^2} = k\ln t\,dt \;\Rightarrow\; \displaystyle\int \frac{dL}{L^2} = \int k\ln t\,dt \;\Rightarrow\; -\frac{1}{L} = kt\ln t - \int k\,dt$ [by parts with $u = \ln t$, $dv = k\,dt$] $\Rightarrow\; -\dfrac{1}{L} = kt\ln t - kt + C \;\Rightarrow\; L = \dfrac{1}{kt - kt\ln t - C}.$

$L(1) = -1 \;\Rightarrow\; -1 = \dfrac{1}{k - k\ln 1 - C} \;\Rightarrow\; C - k = 1 \;\Rightarrow\; C = k + 1$. Thus, $L = \dfrac{1}{kt - kt\ln t - k - 1}.$

15. $\dfrac{dy}{dx} = 4x^3 y,\ y(0) = 7.\quad \dfrac{dy}{y} = 4x^3\,dx$ [if $y \neq 0$] $\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int 4x^3\,dx \;\Rightarrow\; \ln|y| = x^4 + C \;\Rightarrow$

$e^{\ln|y|} = e^{x^4 + C} \;\Rightarrow\; |y| = e^{x^4}e^C \;\Rightarrow\; y = Ae^{x^4};\quad y(0) = 7 \;\Rightarrow\; A = 7 \;\Rightarrow\; y = 7e^{x^4}.$

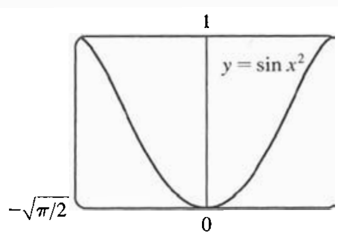
16. $\dfrac{dy}{dx} = \dfrac{y^2}{x^3},\ y(1) = 1.\quad \displaystyle\int \frac{dy}{y^2} = \int \frac{dx}{x^3} \;\Rightarrow\; -\frac{1}{y} = -\frac{1}{2x^2} + C.\quad y(1) = 1 \;\Rightarrow\; -1 = -\frac{1}{2} + C \;\Rightarrow\; C = -\frac{1}{2}.$

So $\dfrac{1}{y} = \dfrac{1}{2x^2} + \dfrac{1}{2} = \dfrac{2 + 2x^2}{2 \cdot 2x^2} \;\Rightarrow\; y = \dfrac{2x^2}{x^2 + 1}.$

17. (a) $y' = 2x\sqrt{1 - y^2} \;\Rightarrow\; \dfrac{dy}{dx} = 2x\sqrt{1-y^2} \;\Rightarrow\; \dfrac{dy}{\sqrt{1-y^2}} = 2x\,dx \;\Rightarrow\; \displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx \;\Rightarrow$

$\sin^{-1} y = x^2 + C$ for $-\frac{\pi}{2} \leq x^2 + C \leq \frac{\pi}{2}.$

(b) $y(0) = 0 \;\Rightarrow\; \sin^{-1} 0 = 0^2 + C \;\Rightarrow\; C = 0$, so $\sin^{-1} y = x^2$ and $y = \sin(x^2)$ for $-\sqrt{\pi/2} \leq x \leq \sqrt{\pi/2}.$

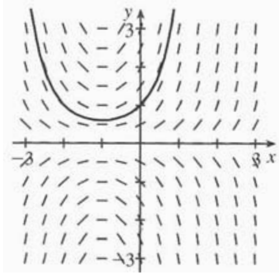


(c) For $\sqrt{1 - y^2}$ to be a real number, we must have $-1 \leq y \leq 1$; that is, $-1 \leq y(0) \leq 1$. Thus, the initial-value problem $y' = 2x\sqrt{1-y^2}$, $y(0) = 2$ does *not* have a solution.

31.

x	y	$y' = y + xy$
0	± 2	± 2
1	± 2	± 4
-3	± 2	∓ 4

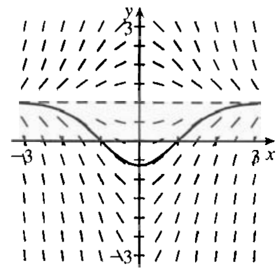
Note that $y' = y(x+1) = 0$ for any point on $y = 0$ or on $x = -1$. The slopes are positive when the factors y and $x+1$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(0, 1)$.



32.

x	y	$y' = x - xy$
± 2	0	± 2
± 2	3	∓ 4
± 2	-1	± 4

Note that $y' = x(1-y) = 0$ for any point on $x = 0$ or on $y = 1$. The slopes are positive when the factors x and $1-y$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(1, 0)$.



33. (a) $\dfrac{dP}{dt} = k(M-P)$ is always positive, so the level of performance P is increasing. As P gets close to M, dP/dt gets close to 0; that is, the performance levels off, as explained in part (a).

(b) $\dfrac{dP}{dt} = k(M-P) \quad \Leftrightarrow \quad \displaystyle\int \frac{dP}{P-M} = \int (-k)\,dt \quad \Leftrightarrow \quad \ln|P-M| = -kt + C \quad \Leftrightarrow \quad |P-M| = e^{-kt+C} \quad \Leftrightarrow$

$P - M = Ae^{-kt}\ \left[A = \pm e^C\right] \quad \Leftrightarrow \quad P = M + Ae^{-kt}$. If we assume that performance is at level 0 when $t = 0$, then

$P(0) = 0 \quad \Leftrightarrow \quad 0 = M + A \quad \Leftrightarrow \quad A = -M \quad \Leftrightarrow \quad P(t) = M - Me^{-kt}. \quad \displaystyle\lim_{t\to\infty} P(t) = M - M \cdot 0 = M$

34. If $S = \dfrac{dT}{dr}$, then $\dfrac{dS}{dr} = \dfrac{d^2T}{dr^2}$. The differential equation $\dfrac{d^2T}{dr^2} + \dfrac{2}{r}\dfrac{dT}{dr} = 0$ can be written as $\dfrac{dS}{dr} + \dfrac{2}{r}S = 0$. Thus,

$\dfrac{dS}{dr} = \dfrac{-2S}{r} \quad \Rightarrow \quad \dfrac{dS}{S} = -\dfrac{2}{r}\,dr \quad \Rightarrow \quad \displaystyle\int \frac{1}{S}\,dS = \int -\frac{2}{r}\,dr \quad \Rightarrow \quad \ln|S| = -2\ln|r| + C$. Assuming $S = dT/dr > 0$

and $r > 0$, we have $S = e^{-2\ln r + C} = e^{\ln r^{-2}}e^C = r^{-2}k\ \left[k = e^C\right] \quad \Rightarrow \quad S = \dfrac{1}{r^2}k \quad \Rightarrow \quad \dfrac{dT}{dr} = \dfrac{1}{r^2}k \quad \Rightarrow$

$dT = \dfrac{1}{r^2}k\,dr \quad \Rightarrow \quad \displaystyle\int dT = \int \frac{1}{r^2}k\,dr \quad \Rightarrow \quad T(r) = -\frac{k}{r} + A.$

$$T(1) = 15 \quad \Rightarrow \quad 15 = -k + A \ \textbf{(1)} \qquad \text{and} \qquad T(2) = 25 \quad \Rightarrow \quad 25 = -\tfrac{1}{2}k + A \ \textbf{(2)}$$

Now solve for k and A: $-2\textbf{(2)} + \textbf{(1)} \quad \Rightarrow \quad -35 = -A$, so $A = 35$ and $k = 20$, and $T(r) = -20/r + 35$.