(b) $y(t) = 4 \times 10^7 \Rightarrow \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \Rightarrow 2 = 1 + 3e^{-0.71t} \Rightarrow e^{-0.71t} = \frac{1}{3} \Rightarrow -0.71t = \ln\frac{1}{3} \Rightarrow$

$t = \dfrac{\ln 3}{0.71} \approx 1.55$ years

39. (a) Our assumption is that $\dfrac{dy}{dt} = ky(1 - y)$, where y is the fraction of the population that has heard the rumor.

(b) The equation in part (a) is the logistic differential equation (7) with $M = 1$, so the solution is given by (8):

$$y = \frac{y_0}{y_0 + (1 - y_0)e^{-kt}}.$$

(c) Let t be the number of hours since 8 AM. Then $y_0 = y(0) = \frac{80}{1000} = 0.08$ and $y(4) = \frac{1}{2}$, so

$\dfrac{1}{2} = y(4) = \dfrac{0.08}{0.08 + 0.92e^{-4k}}$. Thus, $0.08 + 0.92e^{-4k} = 0.16$, $e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23}$, and $e^{-k} = \left(\frac{2}{23}\right)^{1/4}$,

so $y = \dfrac{0.08}{0.08 + 0.92(2/23)^{t/4}} = \dfrac{2}{2 + 23(2/23)^{t/4}}$. Solving this equation for t, we get

$$2y + 23y\left(\frac{2}{23}\right)^{t/4} = 2 \Rightarrow \left(\frac{2}{23}\right)^{t/4} = \frac{2 - 2y}{23y} \Rightarrow \left(\frac{2}{23}\right)^{t/4} = \frac{2}{23} \cdot \frac{1 - y}{y} \Rightarrow \left(\frac{2}{23}\right)^{t/4 - 1} = \frac{1 - y}{y}.$$

It follows that $\dfrac{t}{4} - 1 = \dfrac{\ln[(1 - y)/y]}{\ln\frac{2}{23}}$, so $t = 4\left[1 + \dfrac{\ln((1 - y)/y)}{\ln\frac{2}{23}}\right]$.

When $y = 0.9$, $\dfrac{1 - y}{y} = \frac{1}{9}$, so $t = 4\left(1 - \dfrac{\ln 9}{\ln\frac{2}{23}}\right) \approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 PM.

40. (a) $P(0) = P_0 = 400$, $P(1) = 1200$ and $M = 10{,}000$. From the solution (8) to the logistic differential equation

$P(t) = \dfrac{P_0 M}{P_0 + (M - P_0)e^{-kMt}}$, we get $P = \dfrac{400\,(10{,}000)}{400 + (9600)e^{-kMt}} = \dfrac{10{,}000}{1 + 24e^{-kMt}}$. $P(1) = 1200 \Rightarrow$

$1 + 24e^{-kM} = \frac{100}{12} \Rightarrow e^{kM} = \frac{288}{88} \Rightarrow kM = \ln\frac{36}{11}$. So $P = \dfrac{10{,}000}{1 + 24e^{-t\ln(36/11)}} = \dfrac{10{,}000}{1 + 24 \cdot (11/36)^t}$.

(b) $5000 = \dfrac{10{,}000}{1 + 24(11/36)^t} \Rightarrow 24\left(\frac{11}{36}\right)^t = 1 \Rightarrow t\ln\frac{11}{36} = \ln\frac{1}{24} \Rightarrow t \approx 2.68$ years.

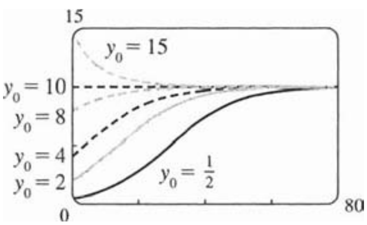
41. (a) $\dfrac{dy}{dt} = ky(M - y) \Rightarrow$

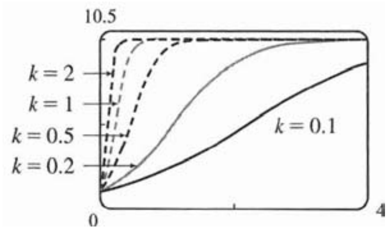
$$\frac{d^2y}{dt^2} = ky\left(-\frac{dy}{dt}\right) + k(M - y)\frac{dy}{dt} = k\frac{dy}{dt}(M - 2y) = k[ky(M - y)](M - 2y) = k^2y(M - y)(M - 2y)$$

(b) y grows fastest when y' has a maximum, that is, when $y'' = 0$. From part (a), $y'' = 0 \Leftrightarrow y = 0, y = M$, or $y = M/2$. Since $0 < y < M$, we see that $y'' = 0 \Leftrightarrow y = M/2$.

42. First we keep k constant (at 0.1, say) and change y_0 in the function $y = \dfrac{10y_0}{y_0 + (10 - y_0)e^{-t}}$. (Notice that y_0 is the y-intercept.) If $y_0 = 0$, the function is 0 everywhere. For $0 < y_0 < 5$, the curve has an inflection point, which moves to the right as y_0 decreases. If $5 < y_0 < 10$, the graph is concave down everywhere. (We are considering only $t \geq 0$.) If $y_0 = 10$, the function is the constant function $y = 10$, and if $y_0 > 10$, the function decreases. For all $y_0 \neq 0$, $\lim_{t \to \infty} y = 10$.

Now we instead keep y_0 constant (at $y_0 = 1$) and change k in the function $y = \dfrac{10}{1 + 9e^{-10kt}}$. It seems that as k increases, the graph approaches the line $y = 10$ more and more quickly. (Note that the only difference in the shape of the curves is in the horizontal scaling; if we choose suitable x-scales, the graphs all look the same.)





43. (a) Let $y(t)$ be the amount of salt (in kg) after t minutes. Then $y(0) = 15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time t (in minutes) is $y(t)/1000$ kg/L and $\dfrac{dy}{dt} = -\left[\dfrac{y(t)}{1000}\,\dfrac{\text{kg}}{\text{L}}\right]\left(10\,\dfrac{\text{L}}{\text{min}}\right) = -\dfrac{y(t)}{100}\,\dfrac{\text{kg}}{\text{min}}$.

$$\int \frac{dy}{y} = -\frac{1}{100}\int dt \;\Rightarrow\; \ln y = -\frac{t}{100} + C, \text{ and } y(0) = 15 \;\Rightarrow\; \ln 15 = C, \text{ so } \ln y = \ln 15 - \frac{t}{100}.$$

It follows that $\ln\left(\dfrac{y}{15}\right) = -\dfrac{t}{100}$ and $\dfrac{y}{15} = e^{-t/100}$, so $y = 15e^{-t/100}$ kg.

(b) After 20 minutes, $y = 15e^{-20/100} = 15e^{-0.2} \approx 12.3$ kg.

44. Let $y(t)$ be the amount of carbon dioxide in the room after t minutes. Then $y(0) = 0.0015(180) = 0.27\ \text{m}^3$. The amount of air in the room is $180\ \text{m}^3$ at all times, so the percentage at time t (in mimutes) is $y(t)/180 \times 100$, and the change in the amount of carbon dioxide with respect to time is

$$\frac{dy}{dt} = (0.0005)\left(2\,\frac{\text{m}^3}{\text{min}}\right) - \frac{y(t)}{180}\left(2\,\frac{\text{m}^3}{\text{min}}\right) = 0.001 - \frac{y}{90} = \frac{9 - 100y}{9000}\,\frac{\text{m}^3}{\text{min}}$$

Hence, $\displaystyle\int \frac{dy}{9 - 100y} = \int \frac{dt}{9000}$ and $-\dfrac{1}{100}\ln|9 - 100y| = \dfrac{1}{9000}t + C$. Because $y(0) = 0.27$, we have $-\frac{1}{100}\ln 18 = C$, so $-\frac{1}{100}\ln|9 - 100y| = \frac{1}{9000}t - \frac{1}{100}\ln 18 \;\Rightarrow\; \ln|9 - 100y| = -\frac{1}{90}t + \ln 18 \;\Rightarrow$ $\ln|9 - 100y| = \ln e^{-t/90} + \ln 18 \;\Rightarrow\; \ln|9 - 100y| = \ln(18e^{-t/90})$, and $|9 - 100y| = 18e^{-t/90}$. Since y is continuous, $y(0) = 0.27$, and the right-hand side is never zero, we deduce that $9 - 100y$ is always negative. Thus, $|9 - 100y| = 100y - 9$ and we have $100y - 9 = 18e^{-t/90} \;\Rightarrow\; 100y = 9 + 18e^{-t/90} \;\Rightarrow\; y = 0.09 + 0.18e^{-t/90}$. The percentage of carbon dioxide in the room is

$$p(t) = \frac{y}{180} \times 100 = \frac{0.09 + 0.18e^{-t/90}}{180} \times 100 = (0.0005 + 0.001e^{-t/90}) \times 100 = 0.05 + 0.1e^{-t/90}$$

In the long run, we have $\lim\limits_{t\to\infty} p(t) = 0.05 + 0.1(0) = 0.05$; that is, the amount of carbon dioxide approaches 0.05% as time goes on.

45. Let $y(t)$ be the amount of alcohol in the vat after t minutes. Then $y(0) = 0.04(500) = 20$ gal. The amount of beer in the vat is 500 gallons at all times, so the percentage at time t (in minutes) is $y(t)/500 \times 100$, and the change in the amount of alcohol with respect to time t is

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} = 0.06\left(5\,\frac{\text{gal}}{\text{min}}\right) - \frac{y(t)}{500}\left(5\,\frac{\text{gal}}{\text{min}}\right) = 0.3 - \frac{y}{100} = \frac{30 - y}{100}\,\frac{\text{gal}}{\text{min}}$$

Hence, $\displaystyle\int \frac{dy}{30 - y} = \int \frac{dt}{100}$ and $-\ln|30 - y| = \frac{1}{100}t + C$. Because $y(0) = 20$, we have $-\ln 10 = C$, so

$-\ln|30-y| = \frac{1}{100}t - \ln 10 \Rightarrow \ln|30-y| = -t/100 + \ln 10 \Rightarrow \ln|30-y| = \ln e^{-t/100} + \ln 10 \Rightarrow$ $\ln|30-y| = \ln(10e^{-t/100}) \Rightarrow |30-y| = 10e^{-t/100}$. Since y is continuous, $y(0) = 20$, and the right-hand side is never zero, we deduce that $30 - y$ is always positive. Thus, $30 - y = 10e^{-t/100} \Rightarrow y = 30 - 10e^{-t/100}$.

The percentage of alcohol is $p(t) = y(t)/500 \times 100 = y(t)/5 = 6 - 2e^{-t/100}$. The percentage of alcohol after one hour is $p(60) = 6 - 2e^{-60/100} \approx 4.9$.

46. (a) If $y(t)$ is the amount of salt (in kg) after t minutes, then $y(0) = 0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\frac{dy}{dt} = \left(0.05\,\frac{\text{kg}}{\text{L}}\right)\left(5\,\frac{\text{L}}{\text{min}}\right) + \left(0.04\,\frac{\text{kg}}{\text{L}}\right)\left(10\,\frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{1000}\,\frac{\text{kg}}{\text{L}}\right)\left(15\,\frac{\text{L}}{\text{min}}\right)$$

$$= 0.25 + 0.40 - 0.015y = 0.65 - 0.015y = \frac{130 - 3y}{200}\,\frac{\text{kg}}{\text{min}}$$

Hence, $\displaystyle\int \frac{dy}{130 - 3y} = \int \frac{dt}{200}$ and $-\frac{1}{3}\ln|130 - 3y| = \frac{1}{200}t + C$. Because $y(0) = 0$, we have $-\frac{1}{3}\ln 130 = C$, so $-\frac{1}{3}\ln|130 - 3y| = \frac{1}{200}t - \frac{1}{3}\ln 130 \Rightarrow \ln|130 - 3y| = -\frac{3}{200}t + \ln 130 = \ln\left(130e^{-3t/200}\right)$, and $|130 - 3y| = 130e^{-3t/200}$. Since y is continuous, $y(0) = 0$, and the right-hand side is never zero, we deduce that $130 - 3y$ is always positive. Thus, $130 - 3y = 130e^{-3t/200}$ and $y = \frac{130}{3}\left(1 - e^{-3t/200}\right)$ kg.

(b) After one hour, $y = \frac{130}{3}\left(1 - e^{-3\cdot 60/200}\right) = \frac{130}{3}\left(1 - e^{-0.9}\right) \approx 25.7$ kg. *Note:* As $t \to \infty$, $y(t) \to \frac{130}{3} = 43\frac{1}{3}$ kg.

47. Assume that the raindrop begins at rest, so that $v(0) = 0$. $dm/dt = km$ and $(mv)' = gm \Rightarrow mv' + vm' = gm \Rightarrow$ $mv' + v(km) = gm \Rightarrow v' + vk = g \Rightarrow \dfrac{dv}{dt} = g - kv \Rightarrow \displaystyle\int \frac{dv}{g - kv} = \int dt \Rightarrow$ $-(1/k)\ln|g - kv| = t + C \Rightarrow \ln|g - kv| = -kt - kC \Rightarrow g - kv = Ae^{-kt}$. $v(0) = 0 \Rightarrow A = g$. So $kv = g - ge^{-kt} \Rightarrow v = (g/k)\left(1 - e^{-kt}\right)$. Since $k > 0$, as $t \to \infty$, $e^{-kt} \to 0$ and therefore, $\lim\limits_{t\to\infty} v(t) = g/k$.

48. (a) $m\dfrac{dv}{dt} = -kv \Rightarrow \dfrac{dv}{v} = -\dfrac{k}{m}\,dt \Rightarrow \ln|v| = -\dfrac{k}{m}t + C$. Since $v(0) = v_0$, $\ln|v_0| = C$.

Therefore, $\ln\left|\dfrac{v}{v_0}\right| = -\dfrac{k}{m}t \Rightarrow \left|\dfrac{v}{v_0}\right| = e^{-kt/m} \Rightarrow v(t) = \pm v_0 e^{-kt/m}$. The sign is + when $t = 0$, and we assume v is continuous, so that the sign is + for all t. Thus, $v(t) = v_0 e^{-kt/m}$.

$ds/dt = v_0 e^{-kt/m} \Rightarrow s(t) = -\dfrac{mv_0}{k}e^{-kt/m} + C'$.

From $s(0) = s_0$, we get $s_0 = -\dfrac{mv_0}{k} + C'$, so $C' = s_0 + \dfrac{mv_0}{k}$ and $s(t) = s_0 + \dfrac{mv_0}{k}\left(1 - e^{-kt/m}\right)$. The distance traveled from time 0 to time t is $s(t) - s_0$, so the total distance traveled is $\lim\limits_{t\to\infty}[s(t) - s_0] = \dfrac{mv_0}{k}$.

Note: In finding the limit, we use the fact that $k > 0$ to conclude that $\lim\limits_{t\to\infty} e^{-kt/m} = 0$.

(b) $m\dfrac{dv}{dt} = -kv^2 \Rightarrow \dfrac{dv}{v^2} = -\dfrac{k}{m}\,dt \Rightarrow \dfrac{-1}{v} = -\dfrac{kt}{m} + C \Rightarrow \dfrac{1}{v} = \dfrac{kt}{m} - C$.

Since $v(0) = v_0$, $C = -\dfrac{1}{v_0}$ and $\dfrac{1}{v} = \dfrac{kt}{m} + \dfrac{1}{v_0}$. Therefore, $v(t) = \dfrac{1}{kt/m + 1/v_0} = \dfrac{mv_0}{kv_0 t + m}$.

$\dfrac{ds}{dt} = \dfrac{mv_0}{kv_0 t + m} \Rightarrow s(t) = \dfrac{m}{k}\displaystyle\int \frac{kv_0\,dt}{kv_0 t + m} = \frac{m}{k}\ln|kv_0 t + m| + C'$. Since $s(0) = s_0$, we get