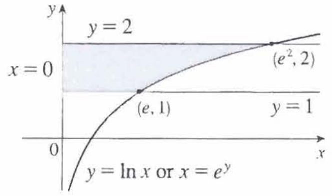
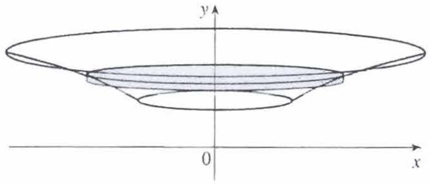
#7.2

4. A cross-section is a disk with radius e^y, so its area is $A(y) = \pi(e^y)^2$.

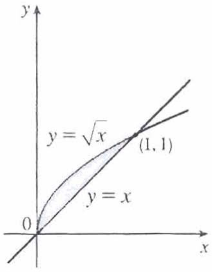
$$V = \int_1^2 \pi(e^y)^2\,dy = \pi\int_1^2 e^{2y}\,dy = \pi\left[\frac{1}{2}e^{2y}\right]_1^2 = \frac{\pi}{2}(e^4 - e^2)$$

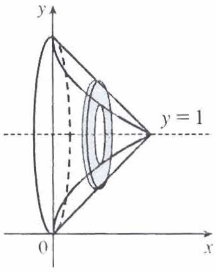




9. A cross-section is a washer with inner radius $1 - \sqrt{x}$ and outer radius $1 - x$, so its area is

$$A(x) = \pi(1-x)^2 - \pi\left(1-\sqrt{x}\right)^2$$
$$= \pi\left[(1 - 2x + x^2) - \left(1 - 2\sqrt{x} + x\right)\right]$$
$$= \pi\left(-3x + x^2 + 2\sqrt{x}\right)$$

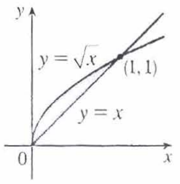


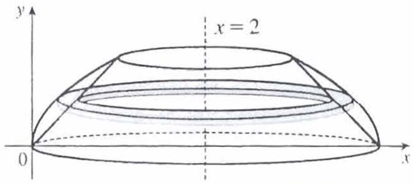


$$V = \int_0^1 A(x)\,dx = \pi\int_0^1\left(-3x + x^2 + 2\sqrt{x}\right)dx = \pi\left[-\tfrac{3}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{4}{3}x^{3/2}\right]_0^1 = \pi\left(-\tfrac{3}{2} + \tfrac{5}{3}\right) = \tfrac{\pi}{6}$$

12. $y = \sqrt{x} \;\Rightarrow\; x = y^2$, so the outer radius is $2 - y^2$.

$$V = \int_0^1 \pi[(2-y^2)^2 - (2-y)^2]\,dy = \pi\int_0^1[(4 - 4y^2 + y^4) - (4 - 4y + y^2)]\,dy$$
$$= \pi\int_0^1 (y^4 - 5y^2 + 4y)\,dy = \pi\left[\tfrac{1}{5}y^5 - \tfrac{5}{3}y^3 + 2y^2\right]_0^1 = \pi\left(\tfrac{1}{5} - \tfrac{5}{3} + 2\right) = \tfrac{8}{15}\pi$$

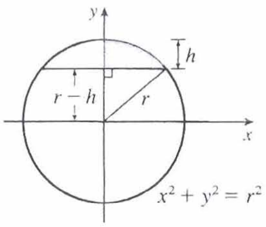




-1.

27. $x^2 + y^2 = r^2 \;\Leftrightarrow\; x^2 = r^2 - y^2$

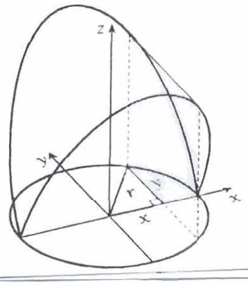
$$V = \pi\int_{r-h}^{r}(r^2 - y^2)\,dy = \pi\left[r^2y - \frac{y^3}{3}\right]_{r-h}^{r}$$
$$= \pi\left\{\left[r^3 - \frac{r^3}{3}\right] - \left[r^2(r-h) - \frac{(r-h)^3}{3}\right]\right\}$$
$$= \pi\left\{\tfrac{2}{3}r^3 - \tfrac{1}{3}(r-h)[3r^2 - (r-h)^2]\right\}$$
$$= \tfrac{1}{3}\pi\left\{2r^3 - (r-h)[3r^2 - (r^2 - 2rh + h^2)]\right\}$$
$$= \tfrac{1}{3}\pi\left\{2r^3 - (r-h)[2r^2 + 2rh - h^2]\right\} = \tfrac{1}{3}\pi(2r^3 - 2r^3 - 2r^2h + rh^2 + 2r^2h + 2rh^2 - h^3)$$
$$= \tfrac{1}{3}\pi(3rh^2 - h^3) = \tfrac{1}{3}\pi h^2(3r - h), \text{ or, equivalently, } \pi h^2\left(r - \frac{h}{3}\right)$$



32. A cross-section is shaded in the diagram.

$A(x) = (2y)^2 = \left(2\sqrt{r^2 - x^2}\right)^2$, so

$$V = \textstyle\int_{-r}^{r} A(x)\,dx = 2\int_0^r 4(r^2 - x^2)\,dx$$
$$= 8\left[r^2x - \tfrac{1}{3}x^3\right]_0^r = 8\left(\tfrac{2}{3}r^3\right) = \tfrac{16}{3}r^3$$



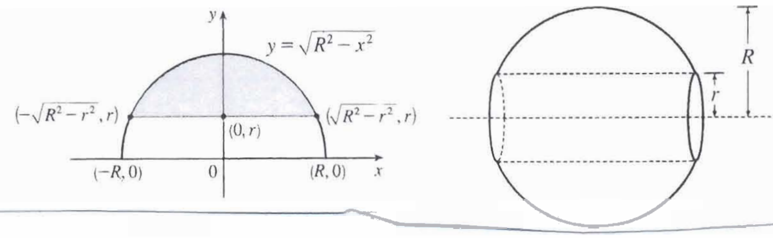
48. The line $y = r$ intersects the semicircle $y = \sqrt{R^2 - x^2}$ when $r = \sqrt{R^2 - x^2}$ $\Rightarrow$ $r^2 = R^2 - x^2$ $\Rightarrow$ $x^2 = R^2 - r^2$ $\Rightarrow$ $x = \pm\sqrt{R^2 - r^2}$. Rotating the shaded region about the x-axis gives us

$$V = \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi\left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2\right] dx = 2\pi \int_0^{\sqrt{R^2-r^2}} (R^2 - x^2 - r^2)\, dx \qquad \text{[by symmetry]}$$

$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left[(R^2 - r^2) - x^2\right] dx = 2\pi\left[(R^2 - r^2)x - \tfrac{1}{3}x^3\right]_0^{\sqrt{R^2-r^2}}$$

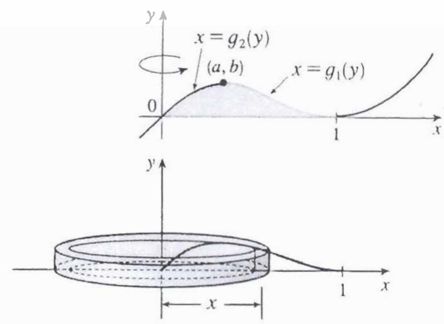
$$= 2\pi\left[(R^2 - r^2)^{3/2} - \tfrac{1}{3}(R^2 - r^2)^{3/2}\right] = 2\pi \cdot \tfrac{2}{3}(R^2 - r^2)^{3/2} = \tfrac{4\pi}{3}(R^2 - r^2)^{3/2}$$

Our answer makes sense in limiting cases. As $r \to 0$, $V \to \frac{4}{3}\pi R^3$, which is the volume of the full sphere. As $r \to R$, $V \to 0$, which makes sense because the hole's radius is approaching that of the sphere.



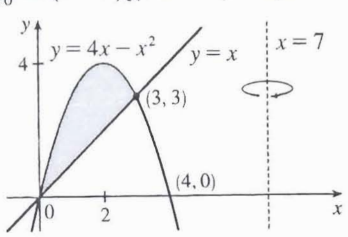
7.3 Volumes by Cylindrical Shells

1. If we were to use the "washer" method, we would first have to locate the local maximum point (a, b) of $y = x(x - 1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x - 1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V = \pi \int_0^b \{[g_1(y)]^2 - [g_2(y)]^2\}\, dy$. Using shells, we find that a typical approximating shell has radius x, so its circumference is $2\pi x$. Its height is y, that is, $x(x - 1)^2$. So the total volume is



$$V = \int_0^1 2\pi x\left[x(x - 1)^2\right] dx = 2\pi \int_0^1 (x^4 - 2x^3 + x^2)\, dx = 2\pi\left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{\pi}{15}$$

22. $V = \int_0^3 2\pi(7 - x)[(4x - x^2) - x]\, dx$



10. Let $u = x^2$. Then $du = 2x\,dx$, so $\int xe^{x^2}\,dx = \int e^u\left(\frac{1}{2}\,du\right) = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$.

14. Let $u = x^2 + 1$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int \frac{x}{(x^2 + 1)^2}\,dx = \int u^{-2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2} \cdot \frac{-1}{u} + C = \frac{-1}{2u} + C = \frac{-1}{2(x^2 + 1)} + C.$$

24. Let $u = \tan^{-1} x$. Then $du = \dfrac{dx}{1 + x^2}$, so $\displaystyle\int \frac{\tan^{-1} x}{1 + x^2}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\tan^{-1} x)^2}{2} + C.$

30. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x+1) + C$.

31. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx = 2\int \frac{\sin x\,\cos x}{1+\cos^2 x}\,dx = 2I$. Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$$2I = -2\int \frac{u\,du}{1+u^2} = -2\cdot\tfrac{1}{2}\ln(1+u^2) + C = -\ln(1+u^2) + C = -\ln(1+\cos^2 x) + C.$$

Or: Let $u = 1 + \cos^2 x$.

32. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\sin x\,dx = -du$, so

$$\int \frac{\sin x}{1+\cos^2 x}\,dx = \int \frac{-du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

33. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so

$$\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = \tan^{-1} x + \int \frac{\frac{1}{2}du}{u} = \tan^{-1} x + \tfrac{1}{2}\ln|u| + C$$
$$= \tan^{-1} x + \tfrac{1}{2}\ln\left|1+x^2\right| + C = \tan^{-1} x + \tfrac{1}{2}\ln\left(1+x^2\right) + C \quad [\text{since } 1+x^2 > 0].$$

34. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{1+x^4}\,dx = \int \frac{\frac{1}{2}\,du}{1+u^2} = \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1}(x^2) + C$.

35. Let $u = x - 1$, so $du = dx$. When $x = 0$, $u = -1$; when $x = 2$, $u = 1$. Thus, $\int_0^2 (x-1)^{25}\,dx = \int_{-1}^1 u^{25}\,du = 0$

by Theorem 7(b), since $f(u) = u^{25}$ is an odd function.

36. Let $u = 4 + 3x$, so $du = 3\,dx$. When $x = 0$, $u = 4$; when $x = 7$, $u = 25$. Thus,

$$\int_0^7 \sqrt{4+3x}\,dx = \int_4^{25} \sqrt{u}\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_4^{25} = \frac{2}{9}(25^{3/2} - 4^{3/2}) = \frac{2}{9}(125 - 8) = \frac{234}{9} = 26.$$

49. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$.

Thus, $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2-1) = 2.$