#6.8

2. (a) Since $y = \dfrac{1}{2x-1}$ is defined and continuous on $[1,2]$, $\displaystyle\int_1^2 \frac{1}{2x-1}\,dx$ is proper.

(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

(c) Since $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)\,dx$ is a Type II improper integral.

8. $\displaystyle\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2}+\frac{1}{2}\right) = \frac{1}{2}\left(0+\frac{1}{2}\right) = \frac{1}{4}.$

Convergent

9. $\int_4^\infty e^{-y/2}\,dy = \lim_{t\to\infty}\int_4^t e^{-y/2}\,dy = \lim_{t\to\infty}\left[-2e^{-y/2}\right]_4^t = \lim_{t\to\infty}(-2e^{-t/2}+2e^{-2}) = 0 + 2e^{-2} = 2e^{-2}.$ Convergent

22. $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx = \lim_{t\to\infty}\int_0^t \frac{e^x}{(e^x)^2+(\sqrt{3})^2}\,dx = \lim_{t\to\infty}\left[\frac{1}{\sqrt{3}}\arctan\frac{e^x}{\sqrt{3}}\right]_0^t = \frac{1}{\sqrt{3}}\lim_{t\to\infty}\left(\arctan\frac{e^t}{\sqrt{3}} - \arctan\frac{1}{\sqrt{3}}\right)$

$\displaystyle = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}\right) = \frac{\pi\sqrt{3}}{9}.$ Convergent

41. For $x \geq 1$, $\dfrac{\cos^2 x}{1+x^2} \leq \dfrac{1}{1+x^2} < \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by (2) with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{\cos^2 x}{1+x^2}\,dx$ is convergent by the Comparison Theorem.

42. For $x \geq 1$, $\dfrac{2+e^{-x}}{x} > \dfrac{2}{x}$ [since $e^{-x} > 0$] $> \dfrac{1}{x}$. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent by (2) with $p = 1 \leq 1$, so $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem.

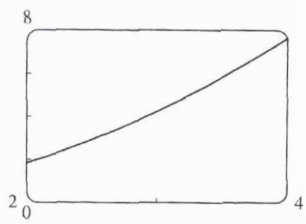
44. For $x \geq 1$, $0 < \dfrac{x}{\sqrt{1+x^6}} < \dfrac{x}{\sqrt{x^6}} = \dfrac{x}{x^3} = \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by (2) with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ is convergent by the Comparison Theorem.

7.4 Arc Length

1. $y = 2 - 3x \;\Rightarrow\; L = \int_{-2}^{1} \sqrt{1 + (dy/dx)^2}\, dx = \int_{-2}^{1} \sqrt{1 + (-3)^2}\, dx = \sqrt{10}\,[1 - (-2)] = 3\sqrt{10}.$

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$L = [\text{distance from } (-2, 8) \text{ to } (1, -1)] = \sqrt{[1 - (-2)]^2 + [(-1) - 8]^2} = \sqrt{90} = 3\sqrt{10}.$

6. 

$$y = \frac{x^2}{2} - \frac{\ln x}{4} \;\Rightarrow\; \frac{dy}{dx} = x - \frac{1}{4x} \;\Rightarrow\; 1 + \left(\frac{dy}{dx}\right)^2 = x^2 + \frac{1}{2} + \frac{1}{16x^2}. \text{ So}$$

$$L = \int_2^4 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{\ln x}{4}\right]_2^4 = \left(8 + \frac{2\ln 2}{4}\right) - \left(2 + \frac{\ln 2}{4}\right) = 6 + \frac{\ln 2}{4}$$

7. $x = \frac{1}{3}\sqrt{y}\,(y - 3) = \frac{1}{3}y^{3/2} - y^{1/2} \;\Rightarrow\; dx/dy = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \;\Rightarrow$

$1 + (dx/dy)^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} = \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2$. So

$$L = \int_1^9 \left(\tfrac{1}{2}y^{1/2} + \tfrac{1}{2}y^{-1/2}\right) dy = \tfrac{1}{2}\left[\tfrac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \tfrac{1}{2}\left[\left(\tfrac{2}{3}\cdot 27 + 2\cdot 3\right) - \left(\tfrac{2}{3}\cdot 1 + 2\cdot 1\right)\right]$$
$$= \tfrac{1}{2}\left(24 - \tfrac{8}{3}\right) = \tfrac{1}{2}\left(\tfrac{64}{3}\right) = \tfrac{32}{3}$$

16. $y = 2^x \;\Rightarrow\; dy/dx = (2^x)\ln 2 \;\Rightarrow\; L = \int_0^3 \sqrt{1 + (\ln 2)^2\, 2^{2x}}\, dx$

#7.5

6. $25 = f(x) = kx = k(0.1)$ $\;[10 \text{ cm} = 0.1 \text{ m}]$, so $k = 250$ N/m and $f(x) = 250x$. Now 5 cm = 0.05 m, so

$W = \int_0^{0.05} 250x\, dx = \left[125x^2\right]_0^{0.05} = 125(0.0025) = 0.3125 \approx 0.31$ J.

12. The work needed to lift the bucket itself is 4 lb · 80 ft = 320 ft-lb. At time t (in seconds) the bucket is $x_i^* = 2t$ ft above its original 80 ft depth, but it now holds only $(40 - 0.2t)$ lb of water. In terms of distance, the bucket holds $\left[40 - 0.2\left(\frac{1}{2}x_i^*\right)\right]$ lb of water when it is x_i^* ft above its original 80 ft depth. Moving this amount of water a distance Δx requires $\left(40 - \frac{1}{10}x_i^*\right)\Delta x$ ft-lb of work. Thus, the work needed to lift the water is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\left(40 - \tfrac{1}{10}x_i^*\right)\Delta x = \int_0^{80}\left(40 - \tfrac{1}{10}x\right)dx = \left[40x - \tfrac{1}{20}x^2\right]_0^{80} = (3200 - 320) \text{ ft-lb}$$

14. The chain's weight density is $\dfrac{25 \text{ lb}}{10 \text{ ft}} = 2.5$ lb/ft. The part of the chain x ft below the ceiling (for $5 \le x \le 10$) has to be lifted $2(x - 5)$ ft, so the work needed to lift the ith subinterval of the chain is $2(x_i^* - 5)(2.5\,\Delta x)$. The total work needed is

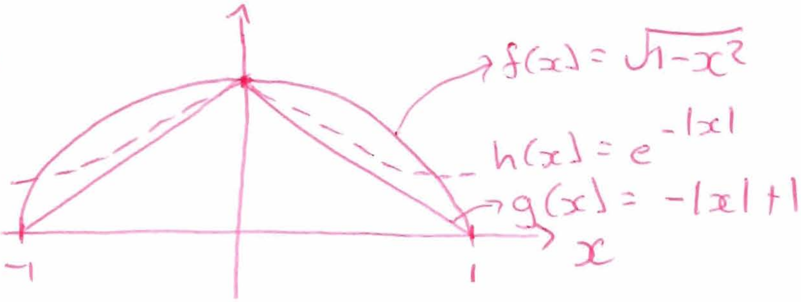
$$W = \lim_{n\to\infty}\sum_{i=1}^{n} 2(x_i^* - 5)(2.5)\,\Delta x = \int_5^{10}[2(x - 5)(2.5)]\,dx = 5\int_5^{10}(x - 5)\,dx$$
$$= 5\left[\tfrac{1}{2}x^2 - 5x\right]_5^{10} = 5\left[(50 - 50) - \left(\tfrac{25}{2} - 25\right)\right] = 5\left(\tfrac{25}{2}\right) = 62.5 \text{ ft-lb}$$

17. (a) A rectangular "slice" of water Δx m thick and lying x ft above the bottom has width x ft and volume $8x\,\Delta x$ m^3. It weighs about $(9.8 \times 1000)(8x\,\Delta x)$ N, and must be lifted $(5 - x)$ m by the pump, so the work needed is about $(9.8 \times 10^3)(5 - x)(8x\,\Delta x)$ J. The total work required is

$$W \approx \int_0^3 (9.8 \times 10^3)(5 - x)8x\,dx = (9.8 \times 10^3)\int_0^3 (40x - 8x^2)\,dx = (9.8 \times 10^3)\left[20x^2 - \tfrac{8}{3}x^3\right]_0^3$$
$$= (9.8 \times 10^3)(180 - 72) = (9.8 \times 10^3)(108) = 1058.4 \times 10^3 \approx 1.06 \times 10^6 \text{ J}$$

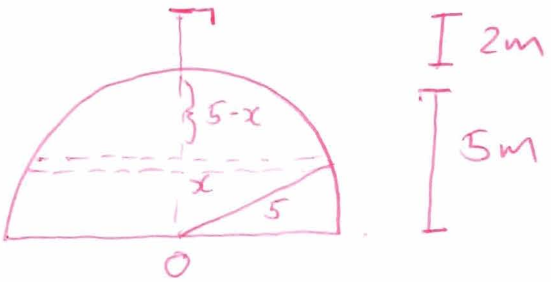
Integration Handout B 15, 17, 18, 26

(15) a)
b)
c)



$g(x) < h(x) < f(x)$

(17)



$Vol(1\ slice) = \pi(\sqrt{5^2-x^2})^2 dx$

$= \pi(25-x^2)dx$

$Weight = 1000\pi g(25-x^2)dx$

$Height = 5-x+2 = 7-x$

$$\int_0^5 1000\pi g(7-x)(25-x^2)dx = \frac{5125}{12} J \times 980\pi$$

$$= 980\pi\left(\frac{5125}{12}\right) J$$

(18) $\int_{-1}^{1} f(x)dx$ = Net # zebra in Seronera region after the months of December & January

(26) $\lim_{t \to \infty} \int_1^t \frac{1}{x^p} dx = \left.\frac{x^{-p+1}}{-p+1}\right|_1^t = \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$

Case I) $p < 1 \Rightarrow \lim_{t \to \infty} \to \infty$ Div

II) $p = 1 \Rightarrow \lim_{t \to \infty} \to \infty$ Div

III) $p > 1 \Rightarrow \lim_{t \to \infty} \to -\frac{1}{1-p}$ Conv