2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$

(b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

10. $a_n = \dfrac{n+1}{3n-1} = \dfrac{1+1/n}{3-1/n}$, so $a_n \to \dfrac{1+0}{3-0} = \dfrac{1}{3}$ as $n \to \infty$. Converges

26. $\displaystyle\lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{H}{=} \lim_{x\to\infty} \frac{2(\ln x)(1/x)}{1} = 2\lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} 2\lim_{x\to\infty} \frac{1/x}{1} = 0$, so by Theorem 3, $\displaystyle\lim_{n\to\infty} \frac{(\ln n)^2}{n} = 0$. Convergent

27. $a_n = \ln(2n^2+1) - \ln(n^2+1) = \ln\left(\dfrac{2n^2+1}{n^2+1}\right) = \ln\left(\dfrac{2+1/n^2}{1+1/n^2}\right) \to \ln 2$ as $n \to \infty$. Convergent

28. $0 < |a_n| = \dfrac{3^n}{n!} = \dfrac{3}{1} \cdot \dfrac{3}{2} \cdot \dfrac{3}{3} \cdot \cdots \cdot \dfrac{3}{(n-1)} \cdot \dfrac{3}{n} \le \dfrac{3}{1} \cdot \dfrac{3}{2} \cdot \dfrac{3}{n}$ [for $n > 2$] $= \dfrac{27}{2n} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\{(-3)^n/n\}$ converges to 0.

29. (a) $a_n = 1000(1.06)^n \Rightarrow a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim_{n\to\infty} a_n = 1000 \lim_{n\to\infty} (1.06)^n$, so the sequence diverges by (8) with $r = 1.06 > 1$.

40. We use induction. Let P_n be the statement that $0 < a_{n+1} \le a_n \le 2$. Clearly P_1 is true, since $a_2 = 1/(3-2) = 1$. Now assume that P_n is true. Then $a_{n+1} \le a_n \Rightarrow -a_{n+1} \ge -a_n \Rightarrow 3 - a_{n+1} \ge 3 - a_n \Rightarrow$ $a_{n+2} = \dfrac{1}{3-a_{n+1}} \le \dfrac{1}{3-a_n} = a_{n+1}$. Also $a_{n+2} > 0$ (since $3 - a_{n+1}$ is positive) and $a_{n+1} \le 2$ by the induction hypothesis, so P_{n+1} is true. To find the limit, we use the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} \Rightarrow L = \frac{1}{3-L} \Rightarrow$ $L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3\pm\sqrt{5}}{2}$. But $L \le 2$, so we must have $L = \frac{3-\sqrt{5}}{2}$.

8.2 Series

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergen

2. $\displaystyle\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5.

In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\displaystyle\sum_{i=1}^{n} a_i$.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{6}{5}$. The series diverges since $|r| = \frac{6}{5} > 1$.

18. $\sum_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with ratio $r = \cos 1 \approx 0.540302$. It converges because $|r| < 1$.

Its sum is $\dfrac{\cos 1}{1 - \cos 1} \approx 1.175343$.

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27. $\sum_{n=1}^{\infty} \dfrac{x^n}{3^n} = \sum_{n=1}^{\infty} \left(\dfrac{x}{3}\right)^n$ is a geometric series with $r = \dfrac{x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x|}{3} < 1$ $\Leftrightarrow$ $|x| < 3$;

that is, $-3 < x < 3$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{x/3}{1 - x/3} = \dfrac{x/3}{1 - x/3} \cdot \dfrac{3}{3} = \dfrac{x}{3-x}$.

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33. (a) The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is

$$S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \frac{D(1-c^n)}{1-c} \text{ by (3).}$$

(b) $$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{D(1-c^n)}{1-c} = \frac{D}{1-c} \lim_{n\to\infty} (1 - c^n) = \frac{D}{1-c} \quad [\text{since } 0 < c < 1 \Rightarrow \lim_{n\to\infty} c^n = 0]$$

$$= \frac{D}{s} \quad [\text{since } c + s = 1] = kD \quad [\text{since } k = 1/s]$$

If $c = 0.8$, then $s = 1 - c = 0.2$ and the multiplier is $k = 1/s = 5$.

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40. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim_{n\to\infty} \dfrac{1}{a_n} \neq 0$, and so $\sum_{n=1}^{\infty} \dfrac{1}{a_n}$ is divergent by the Test for Divergence.

Series Handout

① a) $\sum_{k=1}^{70} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ Not Geometric

b) $\sum_{k=1}^{50} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ Not Geometric

c) $\sum_{k=1}^{60} \frac{1}{k^k} = 1 + \frac{1}{4} + \frac{1}{27} + \dots$ Not Geometric

d) $\sum_{k=1}^{60} (\sqrt[3]{1.01})^k$ $a = \sqrt[3]{1.01}$ $r = \sqrt[3]{1.01}$ $S_{60} = \frac{a(1-r^n)}{1-r} = \frac{\sqrt[3]{1.01}\,(1-1.01^{20/3})}{1-\sqrt[3]{1.01}}$

② a) $a = 1$ $r = \frac{1}{3}$ $S_{100} = \frac{1(1-\frac{1}{3}^{100})}{1-\frac{1}{3}} \sim \frac{3}{2}$

b) $S_\infty = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

c) $a = 1/9$ $r = 1/3$ $S_{100} = \frac{1/9\,(1-(1/3)^{100})}{1-1/3} = \frac{1}{9} \cdot \frac{3}{2} = \frac{1}{6}$

d) $S_\infty = \frac{a}{1-r} = \frac{1/9}{1-1/3} = \frac{1}{6}$

③ No

Yes

④ a) $a = 1$ $r = x$ Converges if $|x| < 1$ to $S_\infty = \frac{1}{1-x}$

b) $a = 1$ $r = x-4$ Converges if $|x-4| < 1 \Rightarrow 5 < x < 7$

(5) a) True

b) True

c) True

d) True

(6) No. Consider example of $b_n = \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \to \infty$