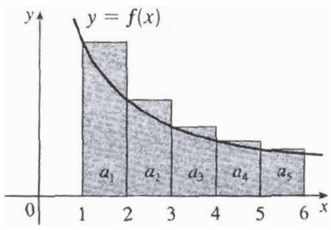
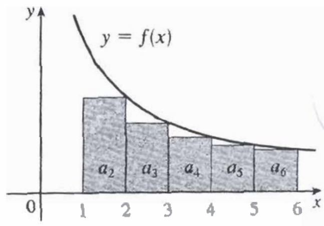
2. From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^5 a_i$. From the second figure, we see that $\sum_{i=2}^6 a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^6 a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^5 a_i$.





5. $\sum\limits_{n=1}^{\infty} n^b$ is a p-series with $p = -b$. $\sum\limits_{n=1}^{\infty} b^n$ is a geometric series. By (1), the p-series is convergent if $p > 1$. In this case, $\sum\limits_{n=1}^{\infty} n^b = \sum\limits_{n=1}^{\infty} (1/n^{-b})$, so $-b > 1 \quad \Leftrightarrow \quad b < -1$ are the values for which the series converge. A geometric series $\sum\limits_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$, so $\sum\limits_{n=1}^{\infty} b^n$ converges if $|b| < 1 \quad \Leftrightarrow \quad -1 < b < 1$.

12. $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^4}$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ are convergent p-series with $p = 4 > 1$ and $p = \frac{3}{2} > 1$, respectively. Thus, $\sum\limits_{n=1}^{\infty} \left(\dfrac{5}{n^4} + \dfrac{4}{n\sqrt{n}}\right) = 5\sum\limits_{n=1}^{\infty} \dfrac{1}{n^4} + 4\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ is convergent by Theorems 8.2.8(i) and 8.2.8(ii).

14. $f(x) = \dfrac{x^2}{x^3+1}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = \dfrac{x(2-x^3)}{(x^3+1)^2} < 0$ for $x \geq 2$, so we can use the Integral Test [note that f is *not* decreasing on $[1, \infty)$].

$\displaystyle\int_2^{\infty} \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty} \left[\tfrac{1}{3}\ln(x^3+1)\right]_2^t = \tfrac{1}{3}\lim_{t\to\infty} \left[\ln(t^3+1) - \ln 9\right] = \infty$, so the series $\sum\limits_{n=2}^{\infty} \dfrac{n^2}{n^3+1}$ diverges, and so does the given series, $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{n^3+1}$.

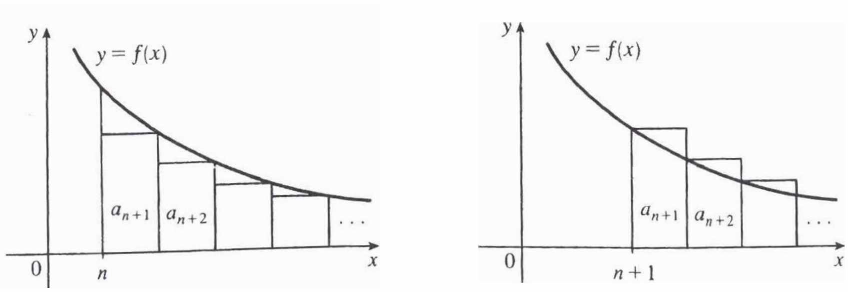
Another solution: Use the Limit Comparison Test with $a_n = \dfrac{n^2}{n^3+1}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 \cdot n}{n^3+1} = \lim_{n\to\infty} \frac{1}{1+1/n^3} = 1 > 0$. Since the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so does $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{n^3+1}$.

16. $\dfrac{n^2-1}{3n^4+1} < \dfrac{n^2}{3n^4+1} < \dfrac{n^2}{3n^4} = \dfrac{1}{3}\dfrac{1}{n^2}$. $\sum\limits_{n=1}^{\infty} \dfrac{n^2-1}{3n^4+1}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p-series [$p = 2 > 1$]. The terms of the given series are positive for $n > 1$, which is good enough.

24. $\dfrac{1+\sin n}{10^n} \leq \dfrac{2}{10^n}$ and $\sum\limits_{n=0}^{\infty} \dfrac{2}{10^n} = 2\sum\limits_{n=0}^{\infty} \left(\dfrac{1}{10}\right)^n$, so the given series converges by comparison with a constant multiple of a convergent geometric series.

29. (a)



We use the same notation and ideas as in the Integral Test, assuming that f is decreasing on $[n, \infty)$. Comparing the areas of the rectangles with the area under $y = f(x)$ for $x > n$ in the first figure, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x)\,dx$$

Similarly, we see from the second figure that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^\infty f(x)\,dx$$

So we have proved that $\int_{n+1}^\infty f(x)\,dx \leq R_n \leq \int_n^\infty f(x)\,dx$.

(b) If we add s_n to each side of the inequalities in part (a), we get $s_n + \int_{n+1}^\infty f(x)\,dx \leq s \leq s_n + \int_n^\infty f(x)\,dx$ because $s_n + R_n = s$.

30. (a) $f(x) = 1/x^4$ is positive and continuous and $f'(x) = -4/x^5$ is negative for $x > 0$, and so the Integral Test applies.

$\sum\limits_{n=1}^{\infty} \dfrac{1}{n^4} \approx s_{10} = \dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \cdots + \dfrac{1}{10^4} \approx 1.082037$. From Exercise 29(a) we have

$R_{10} \leq \displaystyle\int_{10}^\infty \frac{1}{x^4}\,dx = \lim_{t\to\infty}\left[\frac{1}{-3x^3}\right]_{10}^{t} = \lim_{t\to\infty}\left(-\frac{1}{3t^3} + \frac{1}{3\,(10)^3}\right) = \frac{1}{3000}$, so the error is at most $0.000\overline{3}$.

(b) $s_{10} + \displaystyle\int_{11}^\infty \frac{1}{x^4}\,dx \leq s \leq s_{10} + \int_{10}^\infty \frac{1}{x^4}\,dx \;\Rightarrow\; s_{10} + \frac{1}{3(11)^3} \leq s \leq s_{10} + \frac{1}{3(10)^3} \;\Rightarrow$

$1.082037 + 0.000250 = 1.082287 \leq s \leq 1.082037 + 0.000333 = 1.082370$, so we get $s \approx 1.08233$ with error ≤ 0.00005.

(c) $R_n \leq \displaystyle\int_n^\infty \frac{1}{x^4}\,dx = \frac{1}{3n^3}$. So $R_n < 0.00001 \;\Rightarrow\; \dfrac{1}{3n^3} < \dfrac{1}{10^5} \;\Rightarrow\; 3n^3 > 10^5 \;\Rightarrow\; n > \sqrt[3]{(10)^5/3} \approx 32.2$,

that is, for $n > 32$.

41. Since $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N \;\Rightarrow\; 0 \leq a_n < 1$ for all $n > N \;\Rightarrow\; 0 \leq a_n^2 \leq a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

5. $b_n = \dfrac{1}{\sqrt{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

6. $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{\sqrt{n}}{1 + 2\sqrt{n}} = \sum\limits_{n=1}^{\infty} (-1)^n b_n$. Now $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{1}{2 + 1/\sqrt{n}} = \dfrac{1}{2} \neq 0$. Since $\lim\limits_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

8. $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\ln n}{n}\right) = 0 + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{\ln n}{n}\right)$. $b_n = \frac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \frac{\ln x}{x}$, then

$f'(x) = \frac{1 - \ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{x\to\infty}\frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty}\frac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

9. The series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n!} = \sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!}$ satisfies (i) of the Alternating Series Test because

$b_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2\cdot 2^n}{(n+1)n!} = \frac{2}{n+1}\cdot\frac{2^n}{n!} = \frac{2}{n+1}\cdot b_n \leq b_n$ and (ii) $\lim_{n\to\infty}\frac{2^n}{n!} = \frac{2}{n}\cdot\frac{2}{n-1}\cdot\cdots\cdot\frac{2}{2}\cdot\frac{2}{1} = 0$, so the series is convergent. Now $b_7 = 2^7/7! \approx 0.025 > 0.01$ and $b_8 = 2^8/8! \approx 0.006 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 7$. (That is, since the 8th term is less than the desired error, we need to add the first 7 terms to get the sum to the desired accuracy.)

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20. The series $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ has positive terms and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right] = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot\frac{1}{2} = \frac{1}{2} < 1$,

so the series is absolutely convergent by the Ratio Test.

21. $\sum_{n=0}^{\infty}\frac{(-10)^n}{n!}$. Using the Ratio Test, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-10)^n}\right| = \lim_{n\to\infty}\left|\frac{-10}{n+1}\right| = 0 < 1$, so the series is absolutely convergent.

22. $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series: $\lim_{n\to\infty}\frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$. But $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ converges by the Alternating Series Test: $\left\{\frac{n}{n^2+1}\right\}$ has positive terms, is decreasing since $\left(\frac{x}{x^2+1}\right)' = \frac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$, and $\lim_{n\to\infty}\frac{n}{n^2+1} = 0$. Thus, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ is conditionally convergent.

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39. (a) $\lim \left.\frac{1/(n+1)^3}{1/n^3}\right| = \lim_{n\to\infty}\frac{n^3}{(n+1)^3} = \lim_{n\to\infty}\frac{1}{(1+1/n)^3} = 1.$ Inconclusive

$\lim_{n\to\infty}\left|\frac{(n+1)}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \lim_{n\to\infty}\frac{n+1}{2n} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right) = \frac{1}{2}.$ Conclusive (convergent)

(c) $\lim_{n\to\infty}\left|\frac{(-3)^n}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-3)^{n-1}}\right| = 3\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 3\lim_{n\to\infty}\sqrt{\frac{1}{1+1/n}} = 3.$ Conclusive (divergent)

(d) $\lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{1+(n+1)^2}\cdot\frac{1+n^2}{\sqrt{n}}\right| = \lim_{n\to\infty}\left[\sqrt{1+\frac{1}{n}}\cdot\frac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1.$ Inconclusive

Series Handout

(9) a) Limit comparison test:

$\sum \frac{1}{n}$ diverges

Let $a_n = \frac{1}{n+2}$ and $b_n = \frac{1}{n}$

$\Rightarrow \frac{a_n}{b_n} = \frac{\frac{1}{n+2}}{\frac{1}{n}} = \frac{n}{n+2} = \frac{1}{1+2/n}$ $\quad \lim_{n\to\infty} \frac{a_n}{b_n} = 1$

Since $\sum \frac{1}{n}$ diverges so too does $\sum \frac{1}{n+2}$

b) Limit Comparison Test

$a_n = \frac{1}{\sqrt{n^2+10}}$ $\quad b_n = \frac{1}{n}$

$\frac{a_n}{b_n} = \frac{n}{\sqrt{n^2+10}} = \frac{1}{\sqrt{1+10/n^2}}$ $\quad \lim_{n\to\infty} \frac{a_n}{b_n} \to 1$

Since $\sum \frac{1}{n}$ diverges so too does $\sum \frac{1}{\sqrt{10+n^2}}$

c) Integral Test

$\int_2^\infty \frac{1}{x\ln x}\,dx \Rightarrow$ Let $u = \ln x$ $\quad x = 2 \quad u = \ln 2$

$du = \frac{dx}{x}$ $\quad x = \infty \quad u = \infty$

$\Rightarrow \int_{\ln 2}^{\infty} \frac{1}{u}\,du = \ln|u| \Big|_{\ln 2}^{\infty} \to \infty$ Diverges

d) $\int_2^\infty \frac{1}{n(\ln n)^p}\,dn = \int_{\ln 2}^{\infty} \frac{1}{u^p}\,du = \frac{u^{-p+1}}{1-p}\Big|_{\ln 2}^{\infty}$

If $p<1$ the u remains in the numerator
so evaluating at ∞ gives ∞

If $p=1$ the denominator is zero $\to \infty$

If $p>1$ the u goes to the denominator $\to \frac{(\ln 2)^{1-p}}{1-p}$

15) a) AST cannot be invoked since

$$\lim_{k\to\infty} \frac{2k^2-10k}{10k^2+5k} = \lim_{k\to\infty} \frac{2(k-1)}{5(k+1)} \to \frac{2}{5} \neq 0$$

$$\sum_{k=1}^{\infty} (-1)^k \frac{2k^2-10k}{10k^2+5k} = \sum_{k=1}^{\infty} (-1)^k \frac{2}{5}\frac{(k-1)}{(k+1)}$$

$$= \sum_{k=1}^{\infty} (-1)^k \frac{2}{5} - 2(-1)^k \frac{2}{5}\cdot\frac{1}{k+1}$$

$$= \sum_{k=1}^{\infty} (-1)^k \frac{2}{5} - \sum_{k=1}^{\infty} (-1)^k \frac{4}{5(k+1)}$$

$\downarrow$

Does Not Converge

b) By comparison

$$\frac{\sin k}{k^{3/2}} < \frac{1}{k^{3/2}} \quad \forall k$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges, so too must $\sum_{k=1}^{\infty} \frac{\sin k}{k^{3/2}}$

Since $\sum_{k=1}^{\infty} \left|\frac{(-1)^k \sin k}{k^{3/2}}\right|$ converges so too must $\sum_{k=1}^{\infty} \frac{(-1)^k \sin k}{k^{3/2}}$

18) $\sum_{n=0}^{\infty} \frac{(x-3)^n}{5^n}$ $a = 1$, $r = \frac{x-3}{5}$ $\Rightarrow$ $S_\infty = \frac{1}{1-\frac{x-3}{5}} = \frac{5}{2-x}$ if $\boxed{|x-3| < 5}$

Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left(\frac{x-3}{5}\right)^{n+1}}{\left(\frac{x-3}{5}\right)^n} = \frac{x-3}{5}$

Series converges if $\left|\frac{x-3}{5}\right| < 1$ $\Rightarrow$ $\boxed{|x-3| < 5}$

25 a) No The sum to infinity formula is only valid if $|r| < 1$. In this case $r = -1$ so $|r| = 1$

b) It is true that the series does not converge but his use of the summation formula is incorrect. The series diverges because the n^{th} term is $a_n = 3 \cdot 7^{2n}$

So $\lim_{n \to \infty} a_n \to \infty$

c) Not good advice. Consider $\sum_{k=1}^{\infty} \frac{1}{k}$

This is a p-series with $p=1$ so it diverges even though $\lim_{k \to \infty} \frac{1}{k} \to 0$.

Advice: GO TO ANDREW'S SECTION AND LEARN HOW TO DO THIS PROPERLY!!