24. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n\to\infty} \frac{2^n}{n^4} = \infty$, so $\lim_{n\to\infty} (-1)^{n-1} \frac{2^n}{n^4}$ does not exist.

25. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{10^{n+1}}{(n+2)4^{2(n+1)+1}} \cdot \frac{(n+1)4^{2n+1}}{10^n}\right] = \lim_{n\to\infty} \left[\frac{10^{n+1}}{(n+2)4^{2n+3}} \cdot \frac{(n+1)4^{2n+1}}{10^n}\right]$

$$= \lim_{n\to\infty} \left(\frac{10}{4^2} \cdot \frac{n+1}{n+2}\right) = \frac{5}{8} < 1,$$

so the series is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

26. $\left|\frac{\sin 4n}{4^n}\right| \le \frac{1}{4^n}$, so $\sum_{n=1}^{\infty} \left|\frac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ $\left[|r| = \frac{1}{4} < 1\right]$. Thus, $\sum_{n=1} \frac{\sin 4n}{4^n}$ is absolutely convergent.

27. $\frac{|\cos(n\pi/3)|}{n!} \le \frac{1}{n!}$ and $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges (use the Ratio Test), so the series $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ converges absolutely by the Comparison Test.

28. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{5^n/\left[(n+2)^2\, 4^{n+3}\right]}{5^{n-1}/\left[(n+1)^2\, 4^{n+2}\right]} = \frac{5}{4} \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^2 = \frac{5}{4} > 1$, so the series diverges by the Ratio Test.

29. $\left|\frac{(-1)^n \arctan n}{n^2}\right| < \frac{\pi/2}{n^2}$, so since $\sum_{n=1}^{\infty} \frac{\pi/2}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $[p = 2 > 1]$, the given series $\sum_{n=1} \frac{(-1)^n \arctan n}{n^2}$ converges absolutely by the Comparison Test.

30. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$, so the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ converges absolutely by the Root Test.

31. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n^n}{3^{1+3n}}\right)^{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{3}\cdot 3^3} = \infty$, so the series $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ is divergent by the Root Test.

Or: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{3^{4+3n}} \cdot \frac{3^{1+3n}}{n^n}\right] = \lim_{n\to\infty} \left[\frac{1}{3^3} \cdot \left(\frac{n+1}{n}\right)^n (n+1)\right]$

$$= \frac{1}{27} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \lim_{n\to\infty} (n+1) = \frac{1}{27} e \lim_{n\to\infty} (n+1) = \infty,$$

so the series is divergent by the Ratio Test.

32. Since $\left\{\frac{1}{n \ln n}\right\}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by the Alternating Series Test. Since $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the Integral Test (Exercise 8.3.15), the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is conditionally convergent.

37. $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6 \cdot \cdots \cdot (2n)}{n!} = \sum_{n=1}^{\infty} \frac{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot \cdots \cdot (2\cdot n)}{n!} = \sum_{n=1}^{\infty} \frac{2^n n!}{n!} = \sum_{n=1}^{\infty} 2^n$, which diverges by the Test for Divergence since $\lim_{n\to\infty} 2^n = \infty$.

38. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\dfrac{2^{n+1}(n+1)!}{5\cdot 8\cdot 11 \cdot \cdots \cdot (3n+5)}}{\dfrac{2^n n!}{5\cdot 8\cdot 11\cdot \cdots \cdot (3n+2)}}\right| = \lim_{n\to\infty} \frac{2(n+1)}{3n+5} = \frac{2}{3} < 1$, so the series converges absolutely by the Ratio Test.

39. (a) $\lim_{n\to\infty} \left|\frac{1/(n+1)^3}{1/n^3}\right| = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = \lim_{n\to\infty} \frac{1}{(1+1/n)^3} = 1.$ Inconclusive

(b) $\lim_{n\to\infty} \left|\frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n}\right| = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2}.$ Conclusive (convergent)

(c) $\lim_{n\to\infty} \left|\frac{(-3)^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-3)^{n-1}}\right| = 3 \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 3 \lim_{n\to\infty} \sqrt{\frac{1}{1+1/n}} = 3.$ Conclusive (divergent)

(d) $\lim_{n\to\infty} \left|\frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}}\right| = \lim_{n\to\infty} \left[\sqrt{1+\frac{1}{n}} \cdot \frac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1.$ Inconclusive

8. If $a_n = \dfrac{x^n}{n3^n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)3^{n+1}}\cdot\dfrac{n3^n}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{xn}{(n+1)3}\right| = \dfrac{|x|}{3}\lim\limits_{n\to\infty}\dfrac{n}{n+1} = \dfrac{|x|}{3}$. By the Ratio Test, the series converges when $\dfrac{|x|}{3} < 1 \iff |x| < 3$, so $R = 3$. When $x = -3$, the series is the alternating harmonic series, which converges by the Alternating Series Test. When $x = 3$, it is the harmonic series, which diverges. Thus, $I = [-3, 3)$.

10. $a_n = \dfrac{x^n}{5^n n^5}$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{5^{n+1}(n+1)^5}\cdot\dfrac{5^n n^5}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{5}\left(\dfrac{n}{n+1}\right)^5 = \dfrac{|x|}{5}$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{x^n}{5^n n^5}$ converges when $\dfrac{|x|}{5} < 1 \iff |x| < 5$, so $R = 5$. When $x = -5$, we get the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^5}$, which converges by the Alternating Series Test. When $x = 5$, we get the convergent p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^5}$ $[p = 5 > 1]$. Thus, $I = [-5, 5]$.

16. $a_n = \dfrac{n(x-4)^n}{n^3+1}$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-4|^{n+1}}{(n+1)^3+1}\cdot\frac{n^3+1}{n\,|x-4|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{n^3+1}{n^3+3n^2+3n+2}\,|x-4| = |x-4|.$$

By the Ratio Test, the series converges when $|x-4| < 1$ $[\text{so } R = 1] \iff -1 < x-4 < 1 \iff 3 < x < 5$.

When $|x-4| = 1$, $\sum\limits_{n=1}^{\infty}|a_n| = \sum\limits_{n=1}^{\infty}\dfrac{n}{n^3+1}$, which converges by comparison with the convergent p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ $[p = 2 > 1]$. Thus, $I = [3, 5]$.

17. If $a_n = n!(2x-1)^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim\limits_{n\to\infty}(n+1)\,|2x-1| \to \infty$ as $n\to\infty$ for all $x \neq \frac{1}{2}$. Since the series diverges for all $x \neq \frac{1}{2}$, $R = 0$ and $I = \{\frac{1}{2}\}$.

18. $a_n = \dfrac{n^2x^n}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)} = \dfrac{n^2x^n}{2^n n!} = \dfrac{nx^n}{2^n(n-1)!}$, so

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)\,|x|^{n+1}}{2^{n+1}n!}\cdot\dfrac{2^n(n-1)!}{n\,|x|^n} = \lim\limits_{n\to\infty}\dfrac{n+1}{n^2}\dfrac{|x|}{2} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

19. (a) We are given that the power series $\sum\limits_{n=0}^{\infty}c_nx^n$ is convergent for $x = 4$. So by Theorem 3, it must converge for at least $-4 < x \le 4$. In particular, it converges when $x = -2$; that is, $\sum\limits_{n=0}^{\infty}c_n(-2)^n$ is convergent.

(b) It does not follow that $\sum\limits_{n=0}^{\infty}c_n(-4)^n$ is necessarily convergent. [See the comments after Theorem 3 about convergence at the endpoint of an interval. An example is $c_n = (-1)^n/(n4^n)$.]

20. We are given that the power series $\sum\limits_{n=0}^{\infty}c_nx^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \le x < 4$ and diverges for at least $x \ge 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n8^n$ is divergent.

PSET 8 Series Handout Solutions

(12) $f(x) = \dfrac{1}{1-x}$ Taylor Series

$f'(x) = \dfrac{1}{(1-x)^2}$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$f''(x) = \dfrac{2}{(1-x)^3}$

$$= 1 + 1.x + \frac{2}{2!}x^2 + \frac{3.2}{3!}x^3 + \dots$$

$f'''(x) = \dfrac{3.2}{(1-x)^4}$

$$= 1 + x + x^2 + x^3 + \dots$$

$f^{(n)}(x) = \dfrac{n!}{(1-x)^{n+1}}$

$$= \sum_{n=0}^{\infty} x^n$$

(13) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots$$

By Alternating Series Estimation Theorem,

$$b_{N+1} < 10^{-8} \Rightarrow \frac{(0.1)^n}{n!} < 10^{-8} \Rightarrow \frac{n!}{(.1)^n} > 100 \times 10^6$$

$$\Rightarrow \frac{6!}{.1^6} = 720 \times 10^6 > 10^8$$

Hence $b_{N+1} = b_6$ so we need sum to b_5 terms

$$e^{-.1} = 1 - 0.1 + \frac{.1^2}{2} - \frac{.1^3}{3!} + \frac{.1^4}{4!} - \frac{.1^5}{5!}$$

$$= .9048374$$

(14) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

By ASET we need $b_{N+1} < 10^{-6}$

$$\frac{(.2)^7}{7!} < 10^{-6} \Rightarrow b_{N+1} = b_3$$

Sum to $b_2 \Rightarrow b_0 - b_1 + b_2$

$$\Rightarrow \sin(.2) = .2 - \frac{(.2)^3}{3!} + \frac{(.2)^5}{5!}$$

$$= .186669$$

(16) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| \to 0 \quad \forall x$$

(17) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{[2(n+1)]!} \times \frac{(2n)!}{x^{2n}} \right|$

$$= \lim_{n\to\infty} \left| \frac{x^2 \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| \to 0 \quad \forall x$$

(21) Center at $x=1$

Radius of Convergence is 6

(a) $a=1$

(b) Converges for $x=6.5$

Does not converge for $x=-6.5$