

$$\textcircled{11} \quad f(-3) = 0$$

$$f'(x) > 0 \quad -\infty < x < 1$$

$$f'(x) < 0 \quad 1 < x < \infty$$

$$f''(x) < 0 \quad -\infty < x < 2$$

$$f''(x) > 0 \quad 2 < x < \infty$$

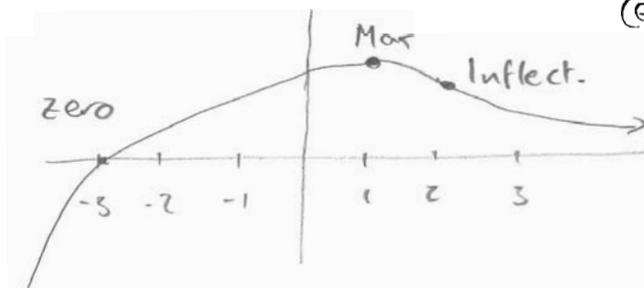
$$(a) \quad a_2 < 0, \quad a_1 > 0, \quad a_0 > 0$$

$$(b) \quad b_2 < 0, \quad b_1 = 0, \quad b_0 > 0$$

$$(c) \quad c_0 > 0, \quad c_1 < 0, \quad c_2 = 0$$

$$(d) \quad c_0 > 0, \quad c_1 < 0, \quad c_2 > 0$$

$$(e) \quad e_0 < 0, \quad e_1 > 0, \quad e_2 < 0$$



$\textcircled{20}$  Series is centered at  $x = 2$

Interval of convergence  $(2-7, 2+7) \Rightarrow (-5, 9)$

$$\textcircled{22} (a) \quad \frac{1}{1-u} = 1 + u + u^2 + \dots$$

$$\int \frac{1}{1-(-u)} du = \int 1 - u + u^2 - u^3 + \dots du$$

$$\ln|1+u| = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n u^{n+1}}{n+1}$$

(b) Let  $u = x-1$

$$\ln|1+u| = \ln|x| = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

$$(c) \quad f(x) = \ln x$$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f'''(x) = \frac{2}{x^3}$$

$$\ln|x| = \ln|1| + 1(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \dots$$

$$= 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

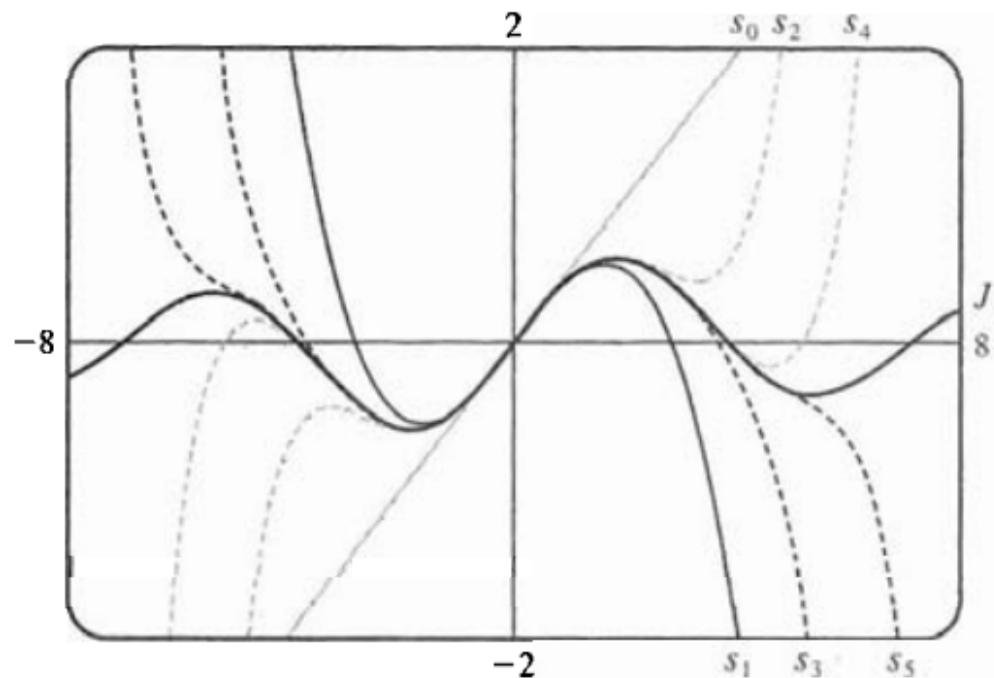
(b), (c) The initial terms of  $J_1(x)$  up to  $n = 5$  are  $a_0 = \frac{x}{2}$ ,

$$a_1 = -\frac{x^3}{16}, a_2 = \frac{x^5}{384}, a_3 = -\frac{x^7}{18,432}, a_4 = \frac{x^9}{1,474,560},$$

$$\text{and } a_5 = -\frac{x^{11}}{176,947,200}.$$

The partial sums seem to

approximate  $J_1(x)$  well near the origin, but as  $|x|$  increases, we need to take a large number of terms to get a good approximation.



24. (a)  $A(x) = 1 + \sum_{n=1}^{\infty} a_n$ , where  $a_n = \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)}$ , so  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|^3 \lim_{n \rightarrow \infty} \frac{1}{(3n+2)(3n+3)} = 0$

for all  $x$ , so the domain is  $\mathbb{R}$ .

6.  $f(x) = \frac{1}{1+9x^2} = \frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty} (-9x^2)^n = \sum_{n=0}^{\infty} (-1)^n 3^{2n} x^{2n}$ . The series converges when  $|-9x^2| < 1$ ; that is, when  $|x| < \frac{1}{3}$ , so  $I = (-\frac{1}{3}, \frac{1}{3})$ .

7.  $f(x) = \frac{1}{x-5} = -\frac{1}{5} \left( \frac{1}{1-x/5} \right) = -\frac{1}{5} \sum_{n=0}^{\infty} \left( \frac{x}{5} \right)^n$  or equivalently,  $-\sum_{n=0}^{\infty} \frac{1}{5^{n+1}} x^n$ . The series converges when  $\left| \frac{x}{5} \right| < 1$ ; that is, when  $|x| < 5$ , so  $I = (-5, 5)$ .

8.  $f(x) = \frac{x}{4x+1} = x \cdot \frac{1}{1-(-4x)} = x \sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-1)^n 2^{2n} x^{n+1}$ . The series converges when  $|-4x| < 1$ ; that is, when  $|x| < \frac{1}{4}$ , so  $I = (-\frac{1}{4}, \frac{1}{4})$ .

9.  $f(x) = \frac{x}{9+x^2} = \frac{x}{9} \left[ \frac{1}{1+(x/3)^2} \right] = \frac{x}{9} \left[ \frac{1}{1-\{-(x/3)^2\}} \right] = \frac{x}{9} \sum_{n=0}^{\infty} \left[ -\left( \frac{x}{3} \right)^2 \right]^n = \frac{x}{9} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{9^n}$   
 $= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{9^{n+1}}$

The geometric series  $\sum_{n=0}^{\infty} \left[ -\left( \frac{x}{3} \right)^2 \right]^n$  converges when  $\left| -\left( \frac{x}{3} \right)^2 \right| < 1 \Leftrightarrow \frac{|x^2|}{9} < 1 \Leftrightarrow |x|^2 < 9 \Leftrightarrow |x| < 3$ , so  $R = 3$  and  $I = (-3, 3)$ .

10.  $f(x) = \frac{x^2}{a^3-x^3} = \frac{x^2}{a^3} \cdot \frac{1}{1-x^3/a^3} = \frac{x^2}{a^3} \sum_{n=0}^{\infty} \left( \frac{x^3}{a^3} \right)^n = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{a^{3n+3}}$ . The series converges when  $|x^3/a^3| < 1 \Leftrightarrow |x^3| < |a^3| \Leftrightarrow |x| < |a|$ , so  $R = |a|$  and  $I = (-|a|, |a|)$ .

11.  $f(x) = \frac{3}{x^2+x-2} = \frac{3}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \Rightarrow 3 = A(x-1) + B(x+2)$ . Taking  $x = -2$ , we get  $A = -1$ . Taking  $x = 1$ , we get  $B = 1$ . Thus,

$$\begin{aligned} \frac{3}{x^2+x-2} &= \frac{1}{x-1} - \frac{1}{x+2} = -\frac{1}{1-x} - \frac{1}{2} \frac{1}{1+x/2} = -\sum_{n=0}^{\infty} x^n - \frac{1}{2} \sum_{n=0}^{\infty} \left( -\frac{x}{2} \right)^n \\ &= \sum_{n=0}^{\infty} \left[ -1 - \frac{1}{2} \left( -\frac{1}{2} \right)^n \right] x^n = \sum_{n=0}^{\infty} \left[ -1 + \left( -\frac{1}{2} \right)^{n+1} \right] x^n = \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1}}{2^{n+1}} - 1 \right] x^n \end{aligned}$$

We represented the given function as the sum of two geometric series; the first converges for  $x \in (-1, 1)$  and the second converges for  $x \in (-2, 2)$ . Thus, the sum converges for  $x \in (-1, 1) = I$ .

12.  $f(x) = \frac{7x-1}{3x^2+2x-1} = \frac{7x-1}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1} = \frac{1}{3x-1} + \frac{2}{x+1} = 2 \cdot \frac{1}{1-(-x)} - \frac{1}{1-3x}$   
 $= 2 \sum_{n=0}^{\infty} (-x)^n - \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} [2(-1)^n - 3^n] x^n$

The series  $\sum (-x)^n$  converges for  $x \in (-1, 1)$  and the series  $\sum (3x)^n$  converges for  $x \in (-\frac{1}{3}, \frac{1}{3})$ , so their sum converges for  $x \in (-\frac{1}{3}, \frac{1}{3}) = I$ .

13. (a)  $f(x) = \frac{1}{(1+x)^2} = \frac{d}{dx} \left( \frac{-1}{1+x} \right) = -\frac{d}{dx} \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right]$  [from Exercise 3]  
 $= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$  [from Theorem 2(i)]  $= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$  with  $R = 1$ .

In the last step, note that we *decreased* the initial value of the summation variable  $n$  by 1, and then *increased* each occurrence of  $n$  in the term by 1 [also note that  $(-1)^{n+2} = (-1)^n$ ].

$$\begin{aligned} \text{(b) } f(x) &= \frac{1}{(1+x)^3} = -\frac{1}{2} \frac{d}{dx} \left[ \frac{1}{(1+x)^2} \right] = -\frac{1}{2} \frac{d}{dx} \left[ \sum_{n=0}^{\infty} (-1)^n (n+1)x^n \right] \quad [\text{from part (a)}] \\ &= -\frac{1}{2} \sum_{n=1}^{\infty} (-1)^n (n+1)n x^{n-1} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1)x^n \quad \text{with } R = 1. \end{aligned}$$

$$\begin{aligned} \text{(c) } f(x) &= \frac{x^2}{(1+x)^3} = x^2 \cdot \frac{1}{(1+x)^3} = x^2 \cdot \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1)x^n \quad [\text{from part (b)}] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1)x^{n+2} \end{aligned}$$

To write the power series with  $x^n$  rather than  $x^{n+2}$ , we will *decrease* each occurrence of  $n$  in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us  $\frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n)(n-1)x^n$ .

$$14. \text{ (a) } \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n \quad [\text{geometric series with } R = 1], \quad \text{so}$$

$$f(x) = \ln(1+x) = \int \frac{dx}{1+x} = \int \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

$[C = 0$  since  $f(0) = \ln 1 = 0]$ , with  $R = 1$

$$\text{(b) } f(x) = x \ln(1+x) = x \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \right] \quad [\text{by part (a)}] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n} = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n-1} \quad \text{with } R = 1.$$

$$\text{(c) } f(x) = \ln(x^2 + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x^2)^n}{n} \quad [\text{by part (a)}] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} \quad \text{with } R = 1.$$

$$15. f(x) = \ln(5-x) = -\int \frac{dx}{5-x} = -\frac{1}{5} \int \frac{dx}{1-x/5} = -\frac{1}{5} \int \left[ \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n \right] dx = C - \frac{1}{5} \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}.$$

Putting  $x = 0$ , we get  $C = \ln 5$ . The series converges for  $|x/5| < 1 \Leftrightarrow |x| < 5$ , so  $R = 5$ .

$$16. \text{ We know that } \frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n. \text{ Differentiating, we get } \frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1} = \sum_{n=0}^{\infty} 2^{n+1} (n+1)x^n, \text{ so}$$

$$f(x) = \frac{x^2}{(1-2x)^2} = \frac{x^2}{2} \cdot \frac{2}{(1-2x)^2} = \frac{x^2}{2} \sum_{n=0}^{\infty} 2^{n+1} (n+1)x^n = \sum_{n=0}^{\infty} 2^n (n+1)x^{n+2} \text{ or } \sum_{n=2}^{\infty} 2^{n-2} (n-1)x^n,$$

with  $R = \frac{1}{2}$ .

$$17. \frac{1}{2-x} = \frac{1}{2(1-x/2)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n \text{ for } \left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2. \text{ Now}$$

$$\frac{1}{(x-2)^2} = \frac{d}{dx} \left( \frac{1}{2-x} \right) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n \right) = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} x^{n-1} = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n. \text{ So}$$

$$f(x) = \frac{x^3}{(x-2)^2} = x^3 \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^{n+3} \text{ or } \sum_{n=3}^{\infty} \frac{n-2}{2^{n-1}} x^n \text{ for } |x| < 2. \text{ Thus, } R = 2 \text{ and } I = (-2, 2).$$

$$18. \text{ From Example 7, } g(x) = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \text{ Thus,}$$

$$\begin{aligned}
 22. f(x) &= \tan^{-1}(2x) = 2 \int \frac{dx}{1+4x^2} = 2 \int \sum_{n=0}^{\infty} (-1)^n (4x^2)^n dx = 2 \int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} dx \\
 &= C + 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \quad [f(0) = \tan^{-1} 0 = 0, \text{ so } C = 0].
 \end{aligned}$$

The series converges when  $|4x^2| < 1 \Leftrightarrow |x| < \frac{1}{2}$ , so

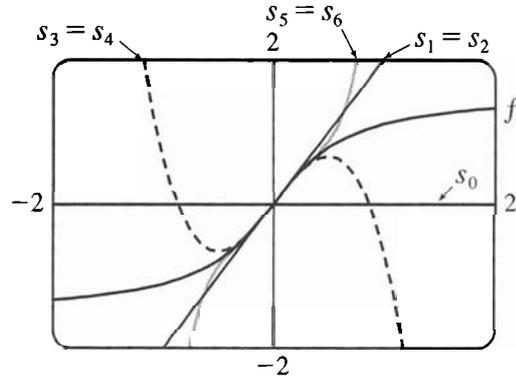
$$R = \frac{1}{2}. \text{ If } x = \pm \frac{1}{2}, \text{ then } f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \text{ and}$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}, \text{ respectively. Both series}$$

converge by the Alternating Series Test. As  $n$  increases,

$s_n(x)$  approximates  $f$  better on the interval of convergence,

which is  $[-\frac{1}{2}, \frac{1}{2}]$ .



$$\begin{aligned}
 23. \frac{t}{1-t^8} &= t \cdot \frac{1}{1-t^8} = t \sum_{n=0}^{\infty} (t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \Rightarrow \int \frac{t}{1-t^8} dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}. \text{ The series for } \frac{1}{1-t^8} \text{ converges} \\
 &\text{when } |t^8| < 1 \Leftrightarrow |t| < 1, \text{ so } R = 1 \text{ for that series and also the series for } t/(1-t^8). \text{ By Theorem 2, the series for} \\
 &\int \frac{t}{1-t^8} dt \text{ also has } R = 1.
 \end{aligned}$$

$$24. \text{ By Example 6, } \ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n} \text{ for } |t| < 1, \text{ so } \frac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty} \frac{t^{n-1}}{n} \text{ and } \int \frac{\ln(1-t)}{t} dt = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}.$$

By Theorem 2,  $R = 1$ .

$$25. \text{ By Example 7, } \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ with } R = 1, \text{ so}$$

$$x - \tan^{-1} x = x - \left( x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1} \text{ and}$$

$$\frac{x - \tan^{-1} x}{x^3} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{2n+1}, \text{ so}$$

$$\int \frac{x - \tan^{-1} x}{x^3} dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{4n^2-1}. \text{ By Theorem 2, } R = 1.$$

$$26. \text{ By Example 7, } \int \tan^{-1}(x^2) dx = \int \sum_{n=0}^{\infty} (-1)^n (x^2)^{2n+1} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)} \text{ with } R = 1.$$

$$27. \frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty} (-x^5)^n = \sum_{n=0}^{\infty} (-1)^n x^{5n} \Rightarrow$$

$$\int \frac{1}{1+x^5} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{5n} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n+1}}{5n+1}. \text{ Thus,}$$

$$I = \int_0^{0.2} \frac{1}{1+x^5} dx = \left[ x - \frac{x^6}{6} + \frac{x^{11}}{11} - \dots \right]_0^{0.2} = 0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \dots. \text{ The series is alternating, so if we use}$$

the first two terms, the error is at most  $(0.2)^{11}/11 \approx 1.9 \times 10^{-9}$ . So  $I \approx 0.2 - (0.2)^6/6 \approx 0.199989$  to six decimal places.

28. From Example 6, we know  $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ , so

$$\ln(1+x^4) = \ln[1 - (-x^4)] = -\sum_{n=1}^{\infty} \frac{(-x^4)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n}}{n} \Rightarrow$$

$$\int \ln(1+x^4) dx = \int \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n}}{n} dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n+1}}{n(4n+1)}. \text{ Thus,}$$

$$I = \int_0^{0.4} \ln(1+x^4) dx = \left[ \frac{x^5}{5} - \frac{x^9}{18} + \frac{x^{13}}{39} - \frac{x^{17}}{68} + \dots \right]_0^{0.4} = \frac{(0.4)^5}{5} - \frac{(0.4)^9}{18} + \frac{(0.4)^{13}}{39} - \frac{(0.4)^{17}}{68} + \dots$$

The series is alternating, so if we use the first three terms, the error is at most  $(0.4)^{17}/68 \approx 2.5 \times 10^{-9}$ .

So  $I \approx (0.4)^5/5 - (0.4)^9/18 + (0.4)^{13}/39 \approx 0.002034$  to six decimal places.

29. We substitute  $3x$  for  $x$  in Example 7, and find that

$$\begin{aligned} \int x \arctan(3x) dx &= \int x \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1} dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+2}}{2n+1} dx \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+3}}{(2n+1)(2n+3)} \end{aligned}$$

$$\begin{aligned} \text{So } \int_0^{0.1} x \arctan(3x) dx &= \left[ \frac{3x^3}{1 \cdot 3} - \frac{3^3 x^5}{3 \cdot 5} + \frac{3^5 x^7}{5 \cdot 7} - \frac{3^7 x^9}{7 \cdot 9} + \dots \right]_0^{0.1} \\ &= \frac{1}{10^3} - \frac{9}{5 \times 10^5} + \frac{243}{35 \times 10^7} - \frac{2187}{63 \times 10^9} + \dots \end{aligned}$$

The series is alternating, so if we use three terms, the error is at most  $\frac{2187}{63 \times 10^9} \approx 3.5 \times 10^{-8}$ . So

$$\int_0^{0.1} x \arctan(3x) dx \approx \frac{1}{10^3} - \frac{9}{5 \times 10^5} + \frac{243}{35 \times 10^7} \approx 0.000983 \text{ to six decimal places.}$$

$$\begin{aligned} 30. \int_0^{0.3} \frac{x^2}{1+x^4} dx &= \int_0^{0.3} x^2 \sum_{n=0}^{\infty} (-1)^n x^{4n} dx = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n x^{4n+3}}{4n+3} \right]_0^{0.3} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{4n+3}}{(4n+3)10^{4n+3}} \\ &= \frac{3^3}{3 \times 10^3} - \frac{3^7}{7 \times 10^7} + \frac{3^{11}}{11 \times 10^{11}} - \dots \end{aligned}$$

The series is alternating, so if we use only two terms, the error is at most  $\frac{3^{11}}{11 \times 10^{11}} \approx 0.00000016$ . So, to six decimal

$$\text{places, } \int_0^{0.3} \frac{x^2}{1+x^4} dx \approx \frac{3^3}{3 \times 10^3} - \frac{3^7}{7 \times 10^7} \approx 0.008969.$$

31. Using the result of Example 6,  $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ , with  $x = -0.1$ , we have

$$\ln 1.1 = \ln[1 - (-0.1)] = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} - \dots. \text{ The series is alternating, so if we use only the}$$

first four terms, the error is at most  $\frac{0.00001}{5} = 0.000002$ . So  $\ln 1.1 \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} \approx 0.09531$ .

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1

We use Equation 7 with  $f(x) = \cos x$ .

$$\begin{aligned}\cos x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\end{aligned}$$

If  $a_n = \frac{(-1)^n x^{2n}}{(2n)!}$ , then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = x^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+1)} \\ &= 0 < 1 \text{ for all } x.\end{aligned}$$

So  $R = \infty$  (Ratio Test).

$f^{(n)}(0) = 0$  if  $n$  is even and  $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$ , so

$$\sin 2x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2^2 |x|^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x.$$

So  $R = \infty$  (Ratio Test).

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2 \cos 2x$	2
2	$-2^2 \sin 2x$	0
3	$-2^3 \cos 2x$	$-2^3$
4	$2^4 \sin 2x$	0

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$e^{5x}$	1
1	$5e^{5x}$	5
2	$5^2 e^{5x}$	25
3	$5^3 e^{5x}$	125
4	$5^4 e^{5x}$	625

$$e^{5x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{5^n}{n!} x^n$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[ \frac{5^{n+1} |x|^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n |x|^n} \right] = \lim_{n \rightarrow \infty} \frac{5|x|}{n+1} \\ &= 0 < 1 \text{ for all } x\end{aligned}$$

So  $R = \infty$ .

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$xe^x$	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3

$$xe^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[ \frac{|x|^{n+1}}{n!} \cdot \frac{(n-1)!}{|x|^n} \right] = \lim_{n \rightarrow \infty} \frac{|x|}{n} \\ &= 0 < 1 \text{ for all } x\end{aligned}$$

So  $R = \infty$ .

23. The general binomial series in (18) is

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots \\(1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} x^3 + \dots \\&= 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{1 \cdot 3 \cdot x^3}{2^3 \cdot 3!} - \frac{1 \cdot 3 \cdot 5 \cdot x^4}{2^4 \cdot 4!} + \dots \\&= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n \cdot n!} \text{ for } |x| < 1, \text{ so } R = 1.\end{aligned}$$

24.  $\frac{1}{(1+x)^4} = (1+x)^{-4} = \sum_{n=0}^{\infty} \binom{-4}{n} x^n$ . The binomial coefficient is

$$\begin{aligned}\binom{-4}{n} &= \frac{(-4)(-5)(-6)\cdots(-4-n+1)}{n!} = \frac{(-4)(-5)(-6)\cdots[-(n+3)]}{n!} \\&= \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots (n+1)(n+2)(n+3)}{2 \cdot 3 \cdot n!} = \frac{(-1)^n (n+1)(n+2)(n+3)}{6}\end{aligned}$$

Thus,  $\frac{1}{(1+x)^4} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)(n+3)}{6} x^n$  for  $|x| < 1$ , so  $R = 1$ .

25.  $\frac{1}{(2+x)^3} = \frac{1}{[2(1+x/2)]^3} = \frac{1}{8} \left(1 + \frac{x}{2}\right)^{-3} = \frac{1}{8} \sum_{n=0}^{\infty} \binom{-3}{n} \left(\frac{x}{2}\right)^n$ . The binomial coefficient is

$$\begin{aligned}\binom{-3}{n} &= \frac{(-3)(-4)(-5)\cdots(-3-n+1)}{n!} = \frac{(-3)(-4)(-5)\cdots[-(n+2)]}{n!} \\&= \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (n+1)(n+2)}{2 \cdot n!} = \frac{(-1)^n (n+1)(n+2)}{2}\end{aligned}$$

Thus,  $\frac{1}{(2+x)^3} = \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)x^n}{2^{n+4}}$  for  $\left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$ , so  $R = 2$ .

$$\begin{aligned}26. (1-x)^{2/3} &= \sum_{n=0}^{\infty} \binom{2/3}{n} (-x)^n = 1 + \frac{2}{3}(-x) + \frac{\frac{2}{3}\left(-\frac{1}{3}\right)}{2!} (-x)^2 + \frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!} (-x)^3 + \dots \\&= 1 - \frac{2}{3}x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} (-1)^n \cdot 2 \cdot [1 \cdot 4 \cdot 7 \cdots (3n-5)]}{3^n \cdot n!} x^n \\&= 1 - \frac{2}{3}x - 2 \sum_{n=2}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3n-5)}{3^n \cdot n!} x^n\end{aligned}$$

and  $|-x| < 1 \Leftrightarrow |x| < 1$ , so  $R = 1$ .

$$27. \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n}}{(2n)!}, \quad R = \infty$$

$$28. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f(x) = e^{-x/2} = \sum_{n=0}^{\infty} \frac{(-x/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^n, \quad R = \infty$$

$$42. \text{ (a) } 1/\sqrt[4]{1+x} = (1+x)^{-1/4} = \sum_{n=0}^{\infty} \binom{-1/4}{n} x^n = 1 - \frac{1}{4}x + \frac{(-1/4)(-5/4)}{2!}x^2 + \frac{(-1/4)(-5/4)(-9/4)}{3!}x^3 + \dots$$

$$= 1 - \frac{1}{4}x + \sum_{n=2}^{\infty} (-1)^n \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{4^n \cdot n!} x^n$$

(b)  $1/\sqrt[4]{1+x} = 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \frac{195}{2048}x^4 - \dots$ .  $1/\sqrt[4]{1.1} = 1/\sqrt[4]{1+0.1}$ , so let  $x = 0.1$ . The sum of the first four terms is then  $1 - \frac{1}{4}(0.1) + \frac{5}{32}(0.1)^2 - \frac{15}{128}(0.1)^3 \approx 0.976$ . The fifth term is  $\frac{195}{2048}(0.1)^4 \approx 0.0000095$ , which does not affect the third decimal place of the sum, so we have  $1/\sqrt[4]{1.1} \approx 0.976$ . (Note that the third decimal place of the sum of the first three terms is affected by the fourth term, so we need to use more than three terms for the sum.)

$$43. \cos x \stackrel{(17)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} \Rightarrow$$

$$x \cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!} \Rightarrow \int x \cos(x^3) dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}, \text{ with } R = \infty.$$

$$44. \frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, \text{ so } \int \frac{\sin x}{x} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

45. Using the series from Exercise 23 and substituting  $x^3$  for  $x$ , we get

$$\int \sqrt{x^3+1} dx = \int \left[ 1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^{3n} \right] dx$$

$$= C + x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n! (3n+1)} x^{3n+1}$$

$$46. e^x \stackrel{(12)}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \Rightarrow \frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \Rightarrow \int \frac{e^x - 1}{x} dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!},$$

with  $R = \infty$ .

47. By Exercise 43,  $\int x \cos(x^3) dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$ , so

$$\int_0^1 x \cos(x^3) dx = \left[ \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+2)(2n)!} = \frac{1}{2} - \frac{1}{8 \cdot 2!} + \frac{1}{14 \cdot 4!} - \frac{1}{20 \cdot 6!} + \dots, \text{ but}$$

$$\frac{1}{20 \cdot 6!} = \frac{1}{14,400} \approx 0.000069, \text{ so } \int_0^1 x \cos(x^3) dx \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{336} \approx 0.440 \text{ (correct to three decimal places) by the}$$

Alternating Series Estimation Theorem.

48. From the table of Maclaurin series in this section, we see that

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ for } x \text{ in } [-1, 1] \text{ and } \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all real numbers } x, \text{ so}$$

$$\tan^{-1}(x^3) + \sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!} \text{ for } x^3 \text{ in } [-1, 1] \Leftrightarrow x \text{ in } [-1, 1]. \text{ Thus,}$$

$$\begin{aligned}
 I &= \int_0^{0.2} [\tan^{-1}(x^3) + \sin(x^3)] dx = \int_0^{0.2} \sum_{n=0}^{\infty} (-1)^n x^{6n+3} \left( \frac{1}{2n+1} + \frac{1}{(2n+1)!} \right) dx \\
 &= \left[ \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{6n+4} \left( \frac{1}{2n+1} + \frac{1}{(2n+1)!} \right) \right]_0^{0.2} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \\
 &= \frac{(0.2)^4}{4}(1+1) - \frac{(0.2)^{10}}{10} \left( \frac{1}{3} + \frac{1}{3!} \right) + \dots
 \end{aligned}$$

But  $\frac{(0.2)^{10}}{10} \left( \frac{1}{3} + \frac{1}{3!} \right) = \frac{(0.2)^{10}}{20} = 5.12 \times 10^{-9}$ , so by the Alternating Series Estimation Theorem,

$I \approx \frac{(0.2)^4}{2} = 0.00080$  (correct to five decimal places). [Actually, the value is 0.0008000, correct to seven decimal places.]

49. We first find a series representation for  $f(x) = (1+x)^{-1/2}$ , and

then substitute.

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{4} \left( \frac{x^2}{2!} \right) - \frac{15}{8} \left( \frac{x^3}{3!} \right) + \dots \Rightarrow$$

$$\frac{1}{\sqrt{1+x^3}} = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \dots \Rightarrow$$

$$\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} = \left[ x - \frac{1}{8}x^4 + \frac{3}{56}x^7 - \frac{1}{32}x^{10} + \dots \right] \approx (0.1) - \frac{1}{8}(0.1)^4, \text{ by the Alternating Series Estimation}$$

Theorem, since  $\frac{3}{56}(0.1)^7 \approx 0.000000054 < 10^{-8}$ , which is the maximum desired error. Therefore,

$$\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} \approx 0.09998750.$$

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-1/2}$	1
1	$-\frac{1}{2}(1+x)^{-3/2}$	$-\frac{1}{2}$
2	$\frac{3}{4}(1+x)^{-5/2}$	$\frac{3}{4}$
3	$-\frac{15}{8}(1+x)^{-7/2}$	$-\frac{15}{8}$
$\vdots$	$\vdots$	$\vdots$

$$50. \int_0^{0.5} x^2 e^{-x^2} dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} dx = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n x^{2n+3}}{n!(2n+3)} \right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)2^{2n+3}}$$

and since the term with  $n=2$  is  $\frac{1}{1792} < 0.001$ , we use  $\sum_{n=0}^1 \frac{(-1)^n}{n!(2n+3)2^{2n+3}} = \frac{1}{24} - \frac{1}{160} \approx 0.0354$ .

$$51. \lim_{x \rightarrow 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x \rightarrow 0} \frac{x - \left( x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \right)}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7 - \dots}{x^3} \\ = \lim_{x \rightarrow 0} \left( \frac{1}{3} - \frac{1}{5}x^2 + \frac{1}{7}x^4 - \dots \right) = \frac{1}{3}$$

since power series are continuous functions.

$$52. \lim_{x \rightarrow 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x \rightarrow 0} \frac{1 - \left( 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \right)}{1 + x - \left( 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \dots \right)} \\ = \lim_{x \rightarrow 0} \frac{\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \dots}{-\frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6 - \dots} \\ = \lim_{x \rightarrow 0} \frac{\frac{1}{2!} - \frac{1}{4!}x^2 + \frac{1}{6!}x^4 - \dots}{-\frac{1}{2!} - \frac{1}{3!}x - \frac{1}{4!}x^2 - \frac{1}{5!}x^3 - \frac{1}{6!}x^4 - \dots} = \frac{\frac{1}{2} - 0}{-\frac{1}{2} - 0} = -1$$

since power series are continuous functions.