Series Handout

8.

$$\sum_{k=1}^{\infty} \frac{1}{2^k+k} = \frac{1}{2+1} + \frac{1}{4+2} + \frac{1}{8+3} + ... + a_n + ...$$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + b_n + ...$$

Now notice that the terms of the first series are always smaller

$$\frac{1}{2+1} < \frac{1}{2},$$

$$\frac{1}{4+2} < \frac{1}{4},$$

$$a_n < b_n.$$

Then $\sum_{k=1}^{\infty} \frac{1}{2^k+k} < \sum_{k=1}^{\infty} \frac{1}{2^k}$, so by that comparison we can say that $\sum_{k=1}^{\infty} \frac{1}{2^k+k}$ converges because $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges.

10. (a) When x is small, the terms $x^3, -4x^2$ is relatively insignificant compared to $4x$. Hence the best linear approximation $P_1(x)$ is

$$P_1(x) = 4x.$$

When x is small, then term x^3 is relatively insignificant compared to $4x$ and $-4x^2$. Therefore the best quadratic approximation $P_2(x)$ should be

$$P_2(x) = 4x - 4x^2.$$

Since $f(x)$ itself is of degree 3, the best cubic approximation $P_3(x)$ must be $f(x)$ itself. I.e.

$$P_3(x) = 4x - 4x^2 + x^3.$$

(b) 2 is a critical point of $f(x)$ implies that the slope, which is also the coefficient a_1, of $f(x)$ at $x = 2$ must be 0. On the other hand,

$$a_k = \frac{f^{(k)}(2)}{k!}.$$

$f(x)$ is a polynomial of degree 3 so $f^{(k)}(x) = 0$ when $k \geq 4$. Thus $a_4, a_5, ..., a_n$ are all zero.

(c) In this case,

$$a_k = \frac{f^{(k)}(b)}{k!}.$$

As in part (b), $a_k = 0$ for $k \geq 4$.

11. (a) At $x = 0$, $f(x)$ is positive, $f(x)$ is increasing and concave down. This tells us that $a_0 > 0$, $a_1 > 0$ and $a_2 < 0$ respectively.

(b) At $x = 1$, $f(x)$ is positive, $x = 1$ is a critical point and $f(x)$ is concave down. Thus $b_0 > 0$, $b_1 = 0$ and $b_2 < 0$.

(c) At $x = 2$, $f(x)$ is positive, $f(x)$ is decreasing and $x = 2$ is an inflection point. Thus $c_0 > 0$, $c_1 < 0$ and $c_2 = 0$.

(d) At $x = 3$, $f(x)$ is positive, $f(x)$ is decreasing and concave up. Thus $d_0 > 0$, $d_1 < 0$ and $d_2 > 0$.

(e) At $x = -4$, $f(x)$ is negative, $f(x)$ is increasing and concave down. Thus $e_0 < 0$, $e_1 > 0$ and $e_2 < 0$.

Supplement 30

1. (a) $f(x) = e^{-x}$. Then $f(0) = e^{-0} = 1$.

$$f'(0) = \frac{d}{dx}e^{-x}\Big|_{x=0} = -e^{-x}\Big|_{x=0} = -1;$$

$$f''(0) = \frac{d}{dx}(-e^{-x})\Big|_{x=0} = e^{-x}\Big|_{x=0} = 1;$$

$$f'''(0) = \frac{d}{dx}e^{-x}\Big|_{x=0} = -e^{-x}\Big|_{x=0} = -1;$$

$$f^{(4)}(0) = \frac{d}{dx}(-e^{-x})\Big|_{x=0} = e^{-x}\Big|_{x=0} = 1.$$

Thus

$$P_1(x) = 1 - x;$$

$$P_2(x) = 1 - x + \frac{1}{2}x^2;$$

$$P_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3;$$

$$P_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4.$$

(c)

$$f(0.1) = e^{-0.1} = 0.90483741...\ ; \quad f(0.3) = e^{-0.3} = 0.740818220...\ .$$

$$P_1(0.1) = 1 - 0.1 = 0.9\ ; \quad P_1(0.3) = 1 - 0.3 = 0.7\ .$$

$$P_2(0.1) = 0.905\ ; \quad P_2(0.3) = 0.745\ .$$

$$P_3(0.1) = 0.90483...\ ; \quad P_3(0.3) = 0.7405\ .$$

$$P_4(0.1) = 0.9048375\ ; \quad P_4(0.3) = 0.7408375\ .$$

2. (a) $f(x) = \ln(1+x)$. Then $f(0) = \ln 1 = 0$.

$$f'(0) = \frac{d}{dx}\ln(1+x)\Big|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1;$$

$$f''(0) = \frac{d}{dx}\left(\frac{1}{1+x}\right)\Big|_{x=0} = \frac{-1}{(1+x)^2}\Big|_{x=0} = -1;$$

$$f'''(0) = \frac{d}{dx}\left(\frac{-1}{(1+x)^2}\right)\Big|_{x=0} = \frac{2}{(1+x)^3}\Big|_{x=0} = 2;$$

$$f^{(4)}(0) = \frac{d}{dx}\left(\frac{2}{(1+x)^3}\right)\Big|_{x=0} = \frac{-6}{(1+x)^4}\Big|_{x=0} = -6.$$

Thus

$$P_1(x) = x;$$

$$P_2(x) = x - \frac{1}{2}x^2;$$

$$P_3(x) = x - \frac{1}{2}x^2 + \frac{2}{3!}x^3 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3;$$

$$P_4(x) = x - \frac{1}{2}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4.$$

(c)

$$\begin{aligned} f(0.1) = \ln 1.1 = 0.0953101\ldots\ ; &\quad f(0.3) = \ln 1.3 = 0.2623642\ldots\ . \\ P_1(0.1) = 0.1\ ; &\quad P_1(0.3) = 0.3\ . \\ P_2(0.1) = 0.095\ ; &\quad P_2(0.3) = 0.255\ . \\ P_3(0.1) = 0.0953\ldots\ ; &\quad P_3(0.3) = 0.264\ . \\ P_4(0.1) = 0.09530833\ldots\ ; &\quad P_4(0.3) = 0.261975\ . \end{aligned}$$

4. (a) $f(x) = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$. Thus

$$P_1(x) = 1 + 4x;$$

$$P_2(x) = 1 + 4x + 6x^2;$$

$$P_3(x) = 1 + 4x + 6x^2 + 4x^3;$$

$$P_4(x) = f(x) = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

(c)

$$\begin{aligned} f(0.1) = 1.4641\ ; &\quad f(0.3) = 2.8561\ . \\ P_1(0.1) = 1.4\ ; &\quad P_1(0.3) = 2.2\ . \\ P_2(0.1) = 1.46\ ; &\quad P_2(0.3) = 2.74\ . \\ P_3(0.1) = 1.464\ ; &\quad P_3(0.3) = 2.848\ . \\ P_4(0.1) = 1.4641\ ; &\quad P_4(0.3) = 2.8561\ . \end{aligned}$$

8. (a) $f(0) < 0$. Thus the constant term of the Taylor polynomial for $f(x)$ at $x = 0$ should be negative.

(b) $f(x)$ is increasing at $x = 0$, i.e. $f'(x) > 0$. Thus the coefficient of x in the Taylor polynomial for $f(x)$ at $x = 0$ should be positive.

(c) $f(x)$ is concave up near $x = 0$, i.e. $f''(x) > 0$. Thus the coefficient of x^2 in the Taylor polynomial for $f(x)$ at $x = 0$ should be positive.

13. We observe (can be proved by mathematical induction!) that, for $k \geq 1$,

$$f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k}.$$

Thus $f(0) = 0$, $f^{(k)}(0) = (-1)^{(k-1)}(k-1)!$ and

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \\ &= \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{k!} x^k \\ &= \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k}. \end{aligned}$$

15. Let $f(x) = \sqrt[3]{x}$. Note $f(8) = \sqrt[3]{8} = 2$ and we consider the Taylor polynomial of $f(x)$ at $x = 8$:

$$f'(8) = \frac{d}{dx}\sqrt[3]{x}\Big|_{x=8} = \frac{1}{3}x^{-\frac{2}{3}}\Big|_{x=8} = \frac{1}{12};$$

$$f''(8) = \frac{d}{dx}(\frac{1}{3}x^{-\frac{2}{3}})\Big|_{x=8} = -\frac{1}{3}\cdot\frac{2}{3}x^{-\frac{5}{3}}\Big|_{x=8} = \frac{-1}{144}.$$

Thus

$$\begin{aligned} P_2(x) &= 2 + \frac{1}{12}(x-8) - \frac{1}{144 \cdot 2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2. \end{aligned}$$

Then $P_2(8.3) = 2.0246875$ gives an approximation of $\sqrt[3]{8.3} = 2.02469385....$

20. For $f(x) = \sqrt{x}$, $f(9) = \sqrt{9} = 3$;

$$f'(9) = \frac{1}{2}x^{-\frac{1}{2}}\Big|_{x=9} = \frac{1}{6};$$

$$f''(9) = \frac{1}{2} \cdot (-\frac{1}{2})x^{-\frac{3}{2}}\Big|_{x=9} = \frac{-1}{108};$$

$$f'''(9) = \frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2})x^{-\frac{5}{2}}\Big|_{x=9} = \frac{1}{648};$$

$$f^{(4)}(9) = \frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdot (-\frac{5}{2})x^{-\frac{7}{2}}\Big|_{x=9} = \frac{-5}{2^4 \cdot 3^6};$$

$$f^{(5)}(9) = \frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdot (-\frac{5}{2}) \cdot (-\frac{7}{2})x^{-\frac{9}{2}}\Big|_{x=9} = \frac{35}{2^5 \cdot 3^8}.$$

So the fifth Taylor polynomial is

$$\begin{aligned} P_5(x) &= 3 + \frac{1}{6}(x-9) + \frac{-1}{108} \cdot \frac{(x-9)^2}{2} + \frac{1}{648} \cdot \frac{(x-9)^3}{3!} + \frac{-5}{2^4 \cdot 3^6} \cdot \frac{(x-9)^4}{4!} + \frac{35}{2^5 \cdot 3^8} \cdot \frac{(x-9)^5}{5!} \\ &= 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3 - \frac{5}{279936}(x-9)^4 + \frac{7}{5038848}(x-9)^5. \end{aligned}$$

∞. Find the Taylor series of $f(x) = \cos x$ at $x = 0$.

For any "good" function $f(x)$, the Taylor series of $f(x)$ at $x = 0$ is given by

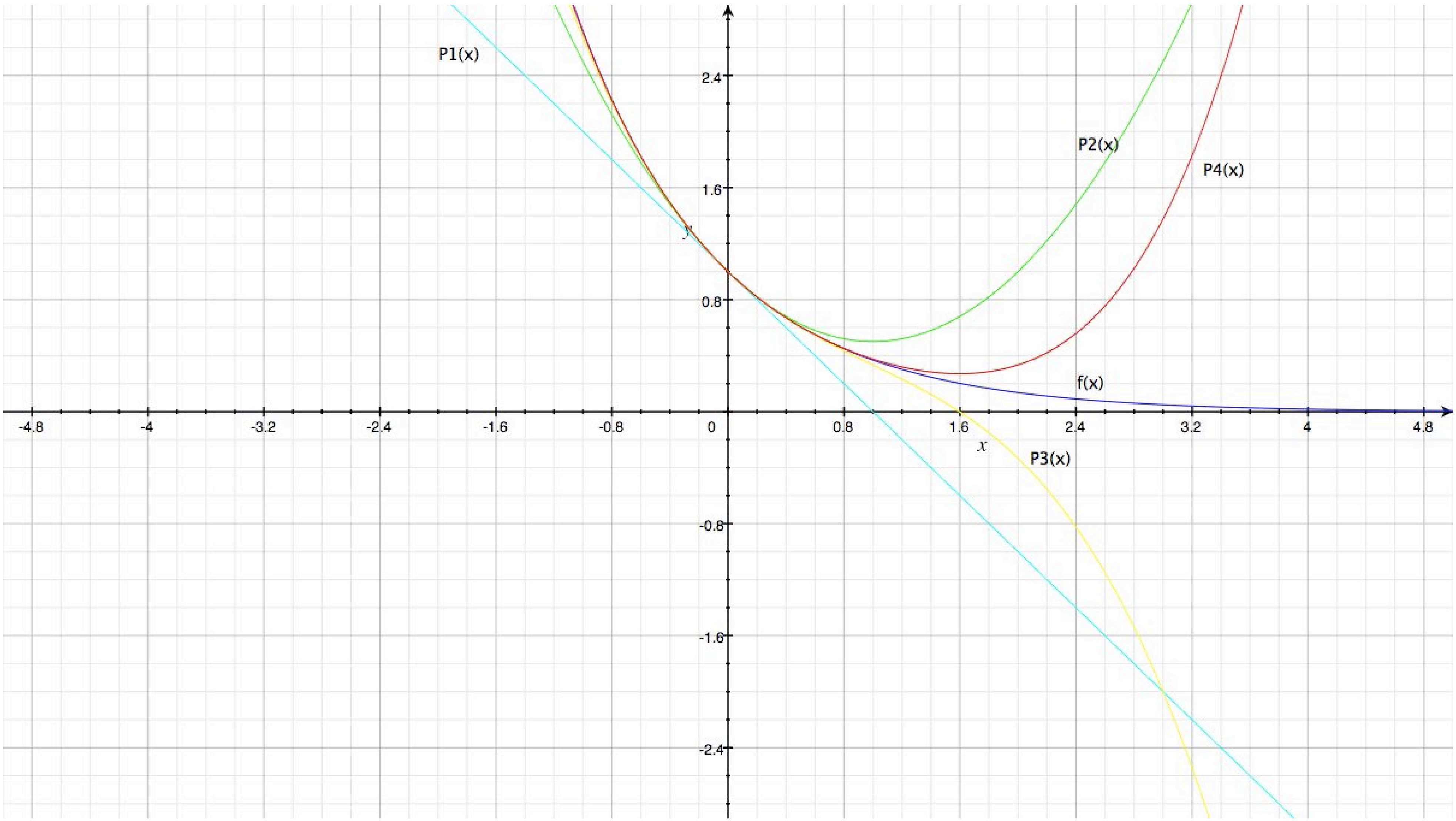
$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}.$$

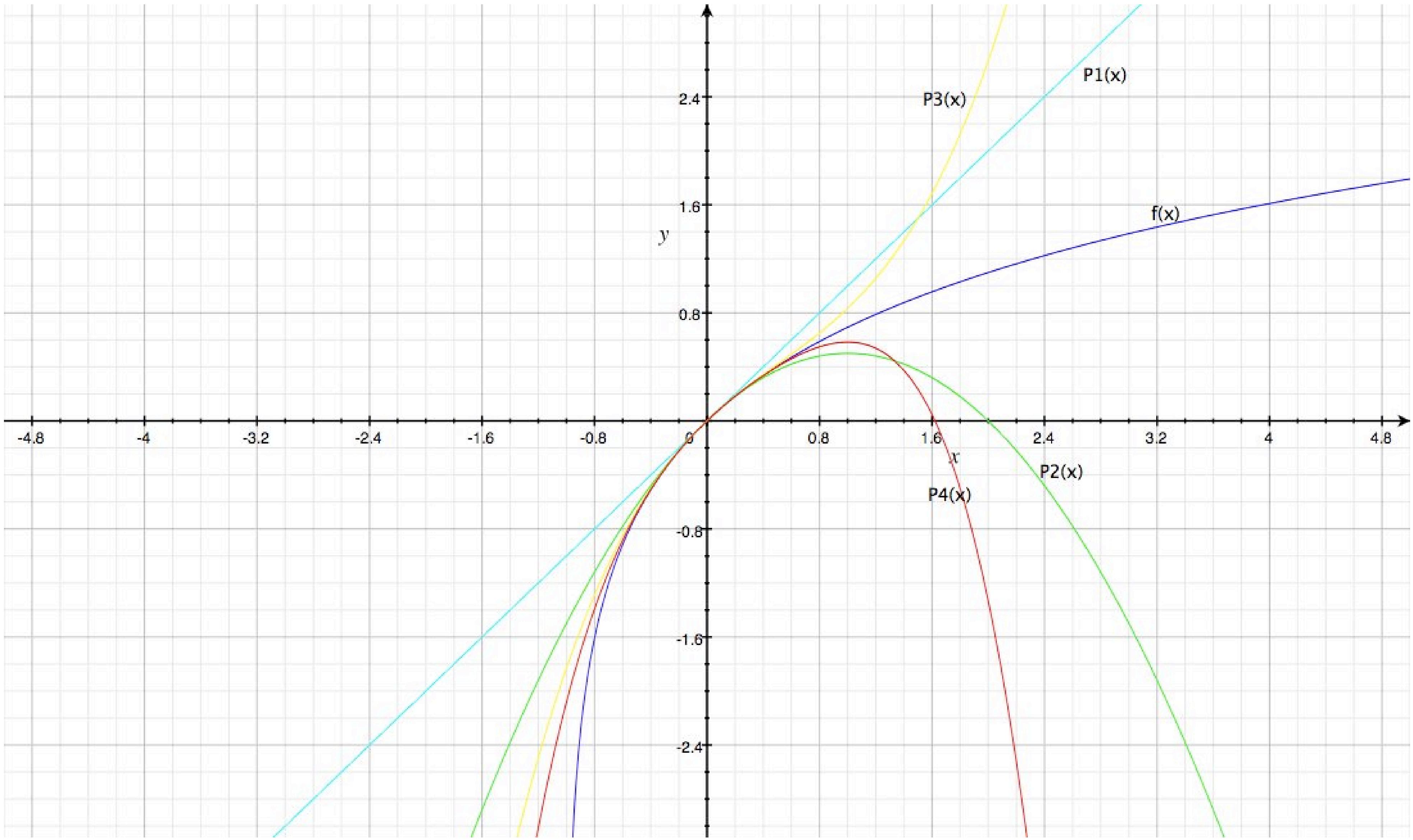
In our case, notice that

$$\begin{aligned} f(x) &= \cos x, \\ f'(x) &= -\sin x, \\ f''(x) &= -\cos x, \\ f'''(x) &= \sin x, \\ f^{(4)} &= \cos x = f(x). \end{aligned}$$

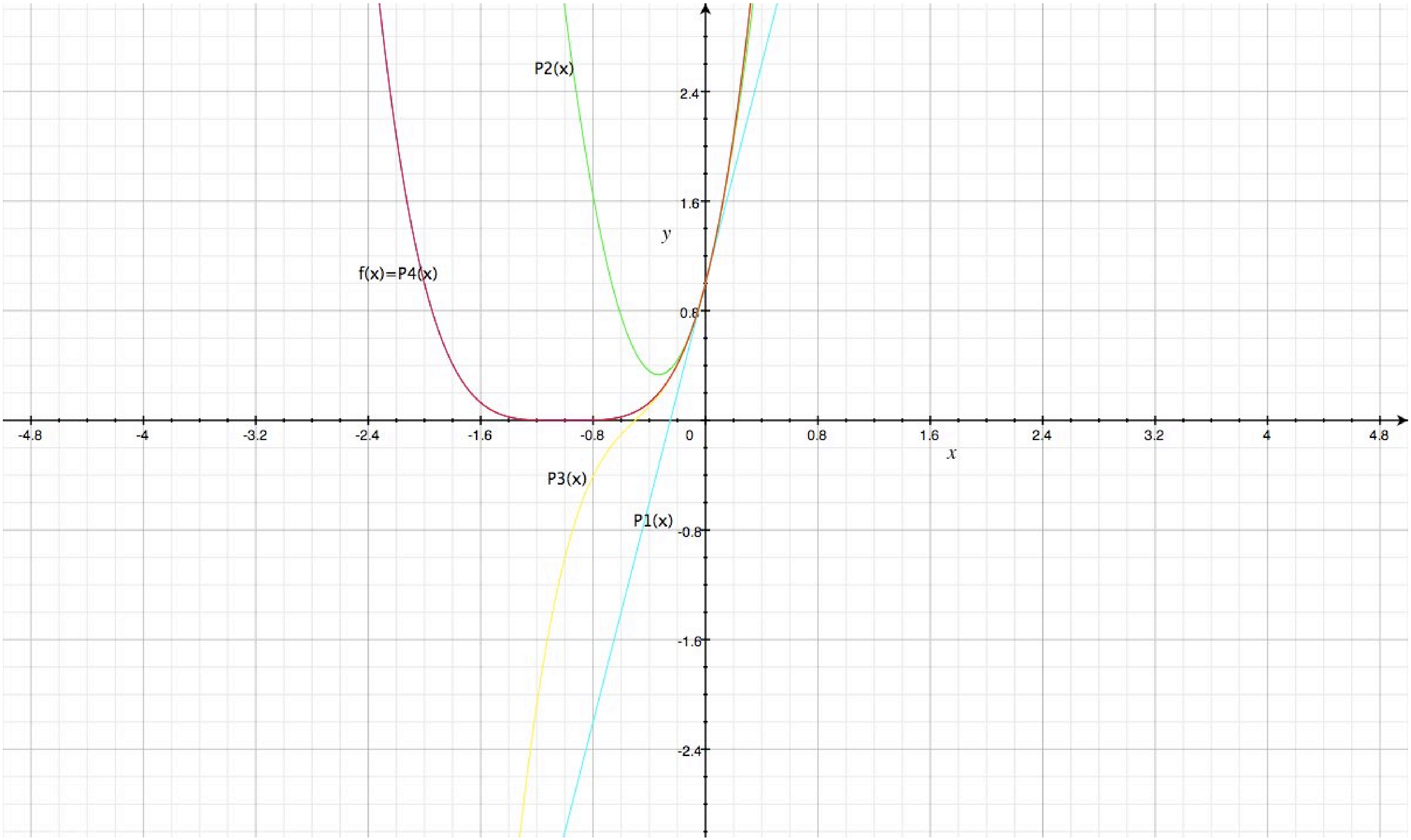
So $f^{(4k)}(x) = \cos x$, $f^{(4k+1)}(x) = -\sin x$, $f^{(4k+2)}(x) = -\cos x$ and $f^{(4k+3)}(x) = \sin x$. Thus $f^{(4k)}(0) = 1$, $f^{(4k+1)}(0) = 0$, $f^{(4k+2)}(0) = -1$, $f^{(4k+3)}(0) = 0$ and the Taylor series of $\cos x$ is

$$\begin{aligned} \cos x &= 1 + (0)x + (-1)\frac{x^2}{2!} + (0)\frac{x^3}{3!} + (1)\frac{x^4}{4!} + (0)\frac{x^5}{5!} + (-1)\frac{x^6}{6!} + (0)\frac{x^7}{7!} + (1)\frac{x^8}{8!} + \cdots \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}. \end{aligned}$$


P1(x)
P2(x)
P3(x)
P4(x)
f(x)
2.4
1.6
0.8
-0.8
-1.6
-2.4
-4.8
-4
-3.2
-2.4
-1.6
-0.8
0
0.8
1.6
2.4
3.2
4
4.8
x
y



P1(x)
P3(x)
f(x)
P2(x)
P4(x)
y
x
2.4
1.6
0.8
-0.8
-1.6
-2.4
-4.8
-4
-3.2
-2.4
-1.6
-0.8
0
0.8
1.6
2.4
3.2
4
4.8



P2(x)
f(x)=P4(x)
P3(x)
P1(x)
y
x
2.4
1.6
0.8
-0.8
-1.6
-2.4
-4.8
-4
-3.2
-2.4
-1.6
-0.8
0
0.8
1.6
2.4
3.2
4
4.8