34. We use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2/[k(n+1)]!}{(n!)^2/(kn)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{[k(n+1)]\,[k(n+1)-1]\cdots[kn+1]}\right|$$

Now if $k = 1$, then this is equal to $\displaystyle\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k = 2$, the limit is

$\displaystyle\lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| = \frac{1}{4} < 1$, so the series converges, and if $k > 2$, then the highest power of n in the denominator is

larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.

35. (a) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = |x|\cdot 0 = 0 < 1$, so by the Ratio Test the

series $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all x.

(b) Since the series of part (a) always converges, we must have $\displaystyle\lim_{n\to\infty}\frac{x^n}{n!} = 0$ by Theorem 8.2.6.

10. $a_n = \dfrac{x^n}{5^n\,n^5}$, so $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+1}(n+1)^5}\cdot\frac{5^n\,n^5}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{5}\left(\frac{n}{n+1}\right)^5 = \frac{|x|}{5}$. By the Ratio Test, the series

$\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{5^n\,n^5}$ converges when $\dfrac{|x|}{5} < 1 \;\Leftrightarrow\; |x| < 5$, so $R = 5$. When $x = -5$, we get the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}$, which

converges by the Alternating Series Test. When $x = 5$, we get the convergent p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^5}$ $(p = 5 > 1)$. Thus,

$I = [-5, 5]$.

14. If $a_n = \dfrac{(-2)^n}{\sqrt{n}}(x+3)^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-2)^n(x+3)^n}\right| = \lim_{n\to\infty}\frac{2\,|x+3|}{\sqrt{1+1/n}} = 2\,|x+3| < 1 \;\Leftrightarrow\; |x+3| < \tfrac{1}{2}$$

$\left[\text{so } R = \tfrac{1}{2}\right] \;\Leftrightarrow\; -\tfrac{7}{2} < x < -\tfrac{5}{2}$. When $x = -\tfrac{7}{2}$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges because it is a p-series with

$p = \tfrac{1}{2} \leq 1$. When $x = -\tfrac{5}{2}$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test. Thus,

$I = \left(-\tfrac{7}{2}, -\tfrac{5}{2}\right]$.

16. $a_n = \dfrac{n(x-4)^n}{n^3+1}$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\left|x-4\right|^{n+1}}{(n+1)^3+1}\cdot\frac{n^3+1}{n\left|x-4\right|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{n^3+1}{n^3+3n^2+3n+2}\left|x-4\right| = \left|x-4\right|.$$

By the Ratio Test, the series converges when $|x-4|<1$ [so $R=1$] $\Leftrightarrow$ $-1<x-4<1$ $\Leftrightarrow$ $3<x<5$.

When $|x-4|=1$, $\displaystyle\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n}{n^3+1}$, which converges by comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$

($p=2>1$). Thus, $I=[3,5]$.

20. We are given that the power series $\sum_{n=0}^{\infty}c_nx^n$ is convergent for $x=-4$ and divergent when $x=6$. So by Theorem 3 it converges for at least $-4\le x<4$ and diverges for at least $x\ge6$ and $x<-6$. Therefore:

(a) It converges when $x=1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x=8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x=-3$; that is, $\sum c_n(-3^n)$ is convergent.

(d) It diverges when $x=-9$; that is, $\sum c_n(-9)^n = \sum(-1)^n c_n 9^n$ is divergent.

28. Since $\sum c_n x^n$ converges whenever $|x|<R$, $\sum c_n x^{2n} = \sum c_n\left(x^2\right)^n$ converges whenever $\left|x^2\right|<R$ $\Leftrightarrow$ $|x|<\sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

6. $f(x) = \dfrac{1}{1+9x^2} = \dfrac{1}{1-(-9x^2)} = \displaystyle\sum_{n=0}^{\infty}(-9x^2)^n = \sum_{n=0}^{\infty}(-1)^n 3^{2n}x^{2n}$. The series converges when $\left|-9x^2\right|<1$; that is, when

$|x|<\frac{1}{3}$, so $I=\left(-\frac{1}{3},\frac{1}{3}\right)$.

11. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}(-1)^n\, x^n\right]$ [from Exercise 3]

$= \sum\limits_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(i)] $= \sum\limits_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R = 1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$= -\dfrac{1}{2}\sum\limits_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \dfrac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R = 1$.

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2 \cdot \dfrac{1}{(1+x)^3} = x^2 \cdot \dfrac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$= \dfrac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$

To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\dfrac{1}{2}\sum\limits_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$.

24. By Example 7, $\displaystyle\int \tan^{-1}(x^2)\,dx = \int \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(x^2)^{2n+1}}{2n+1}\,dx = C + \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{4n+3}}{(2n+1)(4n+3)}$ with $R = 1$.

Series Handout

17. From the power series,

$$a_n = (-1)^n \frac{x^{2n}}{(2n)!}$$

and

$$a_{n+1} = (-1)^{n+1} \frac{x^{2(n+1)}}{(2(n+1))!} = (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!}.$$

Thus

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}} \right| = \lim_{n \to \infty} \left| \frac{x^2 \cdot (2n)!}{(2n+2)!} \right|.$$

Note that $(2n+2)! = (2n)!(2n+1)(2n+2)$ by definition, so we have, for any x,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0.$$

Therefore $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ converges for all x by the Ratio Test.

18. $\sum_{n=0}^{\infty} \frac{(x-3)^n}{5^n}$ is geometric with ratio $r = \frac{(x-3)}{5}$. So the series would converge when $|r| < 1 \Rightarrow -1 < \frac{x-3}{5} < 1 \Rightarrow -5 < x - 3 < 5 \Rightarrow -2 < x < 8$. Now, applying the ratio test, we get:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(x-3)^{n+1}}{5^{n+1}}}{\frac{(x-3)^n}{5^n}} \right| = \lim_{n \to \infty} \left| \frac{x-3}{5} \right| = \left| \frac{x-3}{5} \right|.$$

It converges when

$$\left| \frac{x-3}{5} \right| < 1,$$

exactly what we got by applying what we know about geometric series.

20. A power series of the form $\sum_{k=0}^{\infty} c_k (x-2)^k$ is centered at $x = 2$. A radius of convergence of 7 leaves four possibilities for the interval of convergence, depending on the end points:
- $[-5, 9]$, the series converges at both endpoints.
- $[-5, 9)$, the series converges at $x = -5$ but not at $x = 9$.
- $(-5, 9]$, the series converges at $x = 9$ but not at $x = -5$.
- $(-5, 9)$, the series converges at neither of the end points.

21. (a) Radius of convergence $R = \frac{1}{2}[7 - (-5)] = \frac{1}{2}(12) = 6$. So if $a - 6 = -5$ and $a + 6 = 7$ then $a = 1$.

(b) Since $x = 6.5$ is contained in the interval of convergence, the series converges for $x = 6.5$. Since $x = -6.5$ is NOT contained in the interval of convergence, the series does not converge for $x = -6.5$.

22. (a) Notice that $\frac{d}{du}[\ln(1+u)] = \frac{1}{1+u} = 1 - u + u^2 - u^3 + u^4 - u^5 + \cdots$. Integrate term-by-term, we get

$$\ln(1+u) = C + u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots = C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} u^n}{n}.$$

To determine the constant C, plug in $u = 0$. Then

$$0 = \ln 1 = C.$$

Thus

$$\ln(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} u^n}{n}.$$

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(b) Letting $u = x - 1$, we have

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}.$$

(c) Let $f(x) = \ln x$. Then the power series of $f(x)$ at $x = 1$ is

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(1) \frac{(x-1)^n}{n!}.$$

We know

$$f(x) = \ln x \quad ; \quad f(1) = 0$$
$$f'(x) = \frac{1}{x} \quad ; \quad f'(1) = 1$$
$$f''(x) = \frac{-1}{x^2} \quad ; \quad f''(1) = -1$$
$$f'''(x) = \frac{2}{x^3} \quad ; \quad f'''(1) = 2$$
$$f^{(4)}(x) = \frac{-6}{x^4} \quad ; \quad f^{(4)}(1) = -6,$$

and in general,

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \quad ; \quad f^{(n)}(1) = (-1)^{n+1}(n-1)!.$$

Thus

$$\ln x = 0 + 1 \cdot (x-1) + (-1) \cdot \frac{(x-1)^2}{2} + 2 \cdot \frac{(x-1)^3}{3!} + (-6) \cdot \frac{(x-1)^4}{4!} + \cdots + (-1)^{n+1}(n-1)! \frac{(x-1)^n}{n!} + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}.$$

23. (a) The first three derivatives of $f(x)$ are

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}},$$
$$f''(x) = -\frac{2}{9} x^{-\frac{5}{3}},$$
$$f'''(x) = \frac{10}{27} x^{\frac{8}{3}}.$$

At $a = 27$, $T_3(x) = f(27) + f'(27)(x-27) + \frac{1}{2} f''(27)(x-27)^2 + \frac{1}{6} f'''(27)(x-27)^3$. I.e.

$$T_3(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2 + \frac{5}{531441}(x-27)^3.$$

(b)

$$\sqrt[3]{28} \approx T_3(28) = 3 + \frac{1}{27} - \frac{1}{2187} + \frac{5}{531441} = 3.03658919....$$

(c) An upper bound for the error in this approximation is:

$$E = |a_4| = \left| (-\frac{8}{3})(\frac{10}{27})(27)^{-\frac{11}{3}} \frac{(28-27)^4}{4!} \right|$$

$$= 0.000000232....$$

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(d)

25. (a) A sum cannot be given because this geometric series diverges. It has $r = (-1)$, which is not included in the interval of convergence $-1 < r < 1$.

(b)

$$
\begin{aligned}
S &= 3 + 3 \cdot 7^2 + 3 \cdot 7^4 + \cdots + 3 \cdot 7^{20} \\
(7^2)S &= \quad\; 3 \cdot 7^2 + 3 \cdot 7^4 + \cdots + 3 \cdot 7^{20} + 3 \cdot 7^{22}
\end{aligned}
$$

$$(1 - 7^2)S = 3 - 3 \cdot 7^{22}.$$

$$S = \frac{3(1 - 7^{22})}{1 - 7^2}.$$

Charlie's closed form formula, $\frac{a(1 - r^{n+1})}{1-r}$, is correct. However, he calculated it wrong. The correct expression for the sum is $S = \frac{3(1 - 49^{11})}{1 - 49}$. Charlie's statement about convergence is not correct. If this series were infinite, it would indeed diverge. But the series is finite, so it has a definite sum (that we calculated above).

(c) Charlie is incorrect. The Harmonic Series has $\lim_{n\to\infty} a_n = 0$ and diverges. Amanda is referring to the Nth term test for divergence, which states that if $\lim_{n\to\infty} a_n \neq 0$, the series diverges.