2. For L in equation 4,
(1.5pts)

$$\lambda I - L = \begin{bmatrix} \lambda - a_1 & -a_2 & -a_3 & \cdots & -a_n \\ -b_1 & \lambda & 0 & \cdots & 0 \\ 0 & -b_2 & \lambda & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

Expanding along the first row, we see that each minor is a lower-triangular matrix. So,

$$\det(\lambda I - L) = (\lambda - a_1)\lambda^{n-1} - a_2 b_1 \lambda^{n-2} - a_3 b_1 b_2 \lambda^{n-3} - \cdots - a_n b_1 b_2 \cdots b_{n-1}$$

3a (1.5pts) Let λ_1 be the unique positive eigenvalue of L. Since λ_1 is an eigenvalue of L, it is a root of $g(\lambda) = \det(\lambda I - L)$. Taking the derivative of the characteristic polynomial (derived in 2) and evaluating at λ_1:

$$g'(\lambda_1) = n\lambda^{n-1} - (n-1)a_1\lambda_1^{n-2} - (n-2)a_2 b_1 \lambda^{n-3} - \cdots - a_{n-1}b_1 b_2 \cdots b_{n-2}$$

$$> n\left[\lambda_1^{n-1} - a_1\lambda_1^{n-2} - a_2 b_1 \lambda^{n-3} - \cdots - a_{n-1}b_1 \cdots b_{n-2}\right]$$

(since $a_i \geq 0$ for all i, and $a_i > 0$ for at least one i, and $b_i > 0$ for all i).

$$= \frac{n}{\lambda_1}\left[\lambda_1^{n} - a_1\lambda_1^{n-1} - a_2 b_1 \lambda^{n-2} - \cdots - a_{n-1}b_1 \cdots b_{n-2}\lambda_1\right]$$

$$= \frac{n}{\lambda_1}\left[\cancelto{0}{g(\lambda_1)} + a_n b_1 b_2 \cdots b_{n-1}\right]$$

$$\geq 0 \qquad \text{since } a_n \geq 0,\ b_i > 0 \text{ for all } i.$$

So $g'(\lambda_1) > 0 \Rightarrow g'(\lambda_1) \neq 0 \Rightarrow \lambda_1$ is a simple root.