(10) (1pt)

$$\det\begin{pmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{pmatrix} = 4\begin{vmatrix} 3 & 3 & -1 & 0 \\ 2 & 4 & 2 & 3 \\ 4 & 6 & 2 & 3 \\ 2 & 4 & 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 3 & 3 & 0 \\ 1 & 2 & 4 & 3 \\ 9 & 4 & 6 & 3 \\ 2 & 2 & 4 & 3 \end{vmatrix}$$

$$= -\left(3\begin{vmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 & 3 \\ 9 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 3 \\ 9 & 4 & 3 \\ 2 & 2 & 3 \end{vmatrix}\right)$$

$$= +3\left(\begin{vmatrix} 6 & 3 \\ 4 & 3 \end{vmatrix} - 4\begin{vmatrix} 9 & 3 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 9 & 6 \\ 2 & 4 \end{vmatrix}\right) - 3\left(\begin{vmatrix} 4 & 3 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 9 & 3 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 9 & 4 \\ 2 & 2 \end{vmatrix}\right)$$

$$= 3(6-84+72) - 3(6-42+30)$$

$$= 0$$

(12) (1pt)

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{pmatrix} \qquad \det(A) = -\frac{1}{6} \qquad A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) = -\frac{1}{6}\begin{pmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{pmatrix}$$

(18) (1pt)

$$x - 4y + z = 6$$
$$4x - y + 2z = -1$$
$$2x + 2y - 3z = -20$$

$$A = \begin{pmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix}$$

$$\det(A) = -55$$

$$x = \frac{\begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix}}{\det(A)} \qquad y = \frac{\begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix}}{\det(A)} \qquad z = \frac{\begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix}}{\det(A)}$$

$$x = -\frac{144}{55} \qquad y = -\frac{61}{55} \qquad z = \frac{230}{55}$$

(25) (1pt)

Show: If $\det(A) = 1$, all entries in A are integers $\Longrightarrow$ all entries in A^{-1} are integers.

Proof: If all entries in A are integers, all entries of the adjoint matrix are also integers (Remember the entries of the adjoint matrix are the cofactors of A. The cofactors are simply signed determinants of submatrices of A). From Theorem 2.4.2 we know $A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$. Since $\det(A) = 1$, $A^{-1} = \mathrm{adj}(A)$. So all entries in A^{-1} are integers. $\blacksquare$