Math 20 Fall 2003
Final Exam Practice Problems (Set 1)

1. Consider the following system of linear equations.

$$\begin{array}{rcrcl} x & + & 3y & = & k \\ 4x & + & hy & = & 8 \end{array}$$

   Find values of h and k such that the system...

   (a) ...has no solutions.

   (b) ...has a unique solution.

   (c) ...has infinitely many solutions.

2. Suppose an $n \times n$ matrix A satisfies the equation

$$A^2 - 2A + I = 0.$$

   (a) Express A^3 as a linear combination of A and I.

   (b) Find A^{-1}.

3. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 3 & 2 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

   (a) Find A^{-1}.

   (b) Solve the matrix equation $A\mathbf{x} = \mathbf{b}$.

4. Determine whether each of the following statements is TRUE or FALSE.

   (a) If two rows of a 3×3 matrix A are the same, then $\det(A) = 0$

   (b) If A is a 3×3 matrix, then $\det(5A) = 5\det(A)$.

   (c) If A is $n \times n$ and $\det(A) = 2$, then $\det(A^3) = 6$.

   (d) If B is formed by adding to one row of the square matrix A a linear combination of the other rows, then $\det(B) = \det(A)$.

   (e) If A is a square matrix, then $\det(A^T A) \geq 0$.

5. Compute the following determinant.

$$\begin{vmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{vmatrix}$$

6. Suppose that A is a diagonal matrix with at least one zero main diagonal entry. Show that A cannot be expressed as a product of elementary matrices.

7. Find an equation for the plane that contains the line $x = 3 + 2t$, $y = t$, $z = 8 - t$ and is parallel to the plane $2x + 4y + 8z = 17$.

8. Find the standard matrix for the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that projects a vector onto the line $2x - y = 0$.

9. Suppose that $\mathbf{x}$ is an eigenvector of A corresponding to an eigenvalue λ. Show that $\mathbf{x}$ is an eigenvector of $5I - A$. What is the corresponding eigenvalue?

10. Diagonalize the following matrix, if possible. If not possible, explain why. (You may use the fact that the eigenvalues of the matrix are $\lambda = 1$ and $\lambda = -2$.)

$$\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Answers

1. (a) h must equal 12, k can be any real number except for 2
   (b) h can be any real number except for 12, k can be any real number
   (c) $h = 12$, $k = 2$

2. (a) $A^3 = 3A - 2I$
   (b) $A^{-1} = 2I - A$

3. (a) $A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & -1 & -3 \\ -6 & 1 & 2 \end{bmatrix}$
   (b) $\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

4. (a) True
   (b) False... $\det(5A) = 5^3 \det(A)$
   (c) False... $\det(A^3) = 2^3$
   (d) True
   (e) True

5. 12

6. Since A is a diagonal matrix, $\det(A)$ is equal to the product of the main diagonal entries of A. Since at least one of these entries is 0, the product is 0, and so $\det(A) = 0$. This implies, by the Big Theorem, that A cannot be expressed as a product of elementary matrices.

7. $2x + 4y + 8z = 70$

8. $[T] = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$

9. Since $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, it follows that $A\mathbf{x} = \lambda\mathbf{x}$. Thus, $(5I - A)\mathbf{x} = 5I\mathbf{x} - A\mathbf{x} = 5\mathbf{x} - \lambda\mathbf{x} = (5 - \lambda)\mathbf{x}$. Thus $\mathbf{x}$ is an eigenvector of $(5I - A)$ with eigenvalue $5 - \lambda$.

10. Since the eigenvectors of the eigenvalue $\lambda = 1$ have the form $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} t$, and the eigenvectors of the eigenvalue $\lambda = -2$ have the form $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s$, it follows that the matrix can have at most two linearly independent eigenvectors. Since this 3×3 matrix does not have three linearly independent eigenvectors, it is not diagonalizable by Theorem 7.2.1.