Math 20 Fall 2003
§4.3 Properties of Linear Transformations
November 14, 2003

The Little Theorem: An $n \times n$ matrix A is invertible if and only if the linear system $A\mathbf{x} = \mathbf{w}$ has exactly one solution for every vector $\mathbf{w}$ in $\mathbf{R}^n$ for which the system is consistent.

Proof: Bonus!

Theorem 4.3.1: If A is an $n \times n$ matrix and $T_A : \mathbf{R}^n \to \mathbf{R}^n$ is multiplication by A, then the following statements are equivalent.

1. A is invertible.
2. The range of T_A is $\mathbf{R}^n$.
3. T_A is one-to-one.

Proof: T_A is one-to-one.

$\iff$ For each vector $\mathbf{w}$ in the range of T_A, there is exactly one vector $\mathbf{x}$ in the domain of T_A such that $A\mathbf{x} = \mathbf{w}$.

$\iff$ For each vector $\mathbf{w}$ in $\mathbf{R}^n$ such that $A\mathbf{x} = \mathbf{w}$ for some vector $\mathbf{x}$ in $\mathbf{R}^n$, there is exactly one vector $\mathbf{x}$ in $\mathbf{R}^n$ such that $A\mathbf{x} = \mathbf{w}$.

$\iff$ The system $A\mathbf{x} = \mathbf{w}$ has exactly one solution for every vector $\mathbf{w}$ for which the system has a solution.

$\iff$ A is invertible.

$\iff$ $A\mathbf{x} = \mathbf{w}$ is consistent for every vector $\mathbf{w}$ in $\mathbf{R}^n$.

$\iff$ For every vector $\mathbf{w}$ in $\mathbf{R}^n$, there is some vector $\mathbf{x}$ in $\mathbf{R}^n$ such that $T_A(\mathbf{x}) = \mathbf{w}$.

$\iff$ The range of T_A is $\mathbf{R}^n$.