$\boxed{1.5}$

Problem Set 4

1) a. Elementary, because the matrix can be obtained from I_2 by adding -5 Row 1 to Row 2.

b. Not elementary, because the matrix can be obtained from I_2 only by performing 2 elementary row operations.

c. Elementary, because the matrix can be obtained by multiplying Row 2 of I_2 by $\sqrt{3}$.

d. Elementary, because the matrix can be obtained by interchanging Row 1 and Row 3 of I_3.

e. Not an elementary matrix, because it is not invertible.

f. Elementary matrix, because it can be obtained by adding 9 Row 3 to Row 2 of I_3.

g. Not an elementary matrix, because it can be obtained from I_4 only by performing 2 operations on I_4.

3) a. If we interchange Rows 1 and 3 of A, we get B. Therefore, E_1 must be the matrix obtained from I_3 by interchanging Rows 1 and 3 of I_3

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

b. E_2 can be obtained from I_3 by interchanging Rows 1 and 3.

$$E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

c. E_3 can be obtained from I_3 by replacing its 3rd Row by -2 Row 1 plus Row 3.

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

d. E_4 can be obtained from I_3 by replacing its 3rd Row by 2Row 1 plus Row 3.

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

5) a) $\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right]$

$\xrightarrow[-4\text{Row}2+\text{Row}1 \to \text{Row}1]{(-1)\text{Row}2,\text{ then}} \left[\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right]$ $\quad \boxed{A^{-1} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}}$

c) $\left[\begin{array}{cc|cc} 6 & -4 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{6}R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & -2/3 & 1/6 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right]$

$\xrightarrow{3R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & -2/3 & 1/6 & 0 \\ 0 & 0 & 1/2 & 1 \end{array}\right]$

Not invertible, because we got a row of zeros on the left side.

6.) a) $\left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \text{ and } R_2]{\text{Interchange}} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow[-2R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right] \xrightarrow{-1R_2+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \end{array}\right]$

$\xrightarrow[\text{interchange } R_2 \text{ and } R_3]{-4R_3+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{array}\right] \xrightarrow[-3R_3+R_1 \to R_1]{-\frac{1}{10}R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -11/10 & -6/5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right]$

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

b.) $\left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-4R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -4 & 0 & 1 \end{array}\right]$

Not invertible, because we get a row of zeros.

$\boxed{11.8}$ 1.) a) From Eq. 2, the expected payoff of the game:

$$pAq = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -4 & 6 & -4 & 1 \\ 5 & -7 & 3 & 8 \\ -8 & 0 & 6 & -2 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = -\frac{5}{8}$$

b) If player R uses strategy $[p_1 \ p_2 \ p_3]$ against player C's strategy $\left[\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4}\right]$ his payoff will be $pAq = (-1/4)p_1 + (9/4)p_2 - p_3$.

Since p_1, p_2 and p_3 are nonnegative and add up to 1, this is a weighted average of the numbers $-1/4$, $9/4$ and -1. Clearly this is the largest if $p_1 = p_3 = 0$ and $p_2 = 1$; that is $p = [0 \ 1 \ 0]$

c) As in b), if player C uses $[q_1 \ q_2 \ q_3 \ q_4]$ against $\left[\frac{1}{2} \ 0 \ \frac{1}{2}\right]$, we get $pAq = -6q_1 + 3q_2 + q_3 - \frac{1}{2}q_4$.

Clearly this is minimized over all strategies by setting $q_1 = 1$ and $q_2 = q_3 = q_4 = 0$. That is $q = [1 \ 0 \ 0 \ 0]$

3. a) a_{22} – saddle point, so optimal strategies are pure;
$p = [0 \ 1]$ $q = [0 \ 1]$, the value of the game is $a_{22} = 3$

b) a_{21} – saddle point $\Rightarrow$ $p = [0 \ 1 \ 0]$ $q = [1 \ 0]$
The value of the game is $a_{21} = 2$

c) a_{32} – saddle point $\Rightarrow$ $p = [0 \ 0 \ 1]$ $q = [0 \ 1 \ 0]$
$v = a_{32} = 2$

d) a_{21} – saddle point $\Rightarrow$ $p = [0 \ 1 \ 0 \ 0]$ $q = [1 \ 0 \ 0]$
$v = a_{21} = -2$

5. Let a_{11} = payoff to R if the black ace and black 2 are played $= 3$
a_{12} = payoff to R if the black ase and red 3 are played $= -4$
a_{21} = payoff to R if the ~~black~~ red 4 and black 2 are played $= -6$
a_{22} = payoff to R if the red 4 and red 3 are played $= 7$

So, the payoff matrix for the game is $A = \begin{bmatrix} 3 & -4 \\ -6 & 7 \end{bmatrix}$

A has no saddle points, so from Theorem 2,
$$p = \begin{bmatrix} \frac{13}{20} & \frac{7}{20} \end{bmatrix}, \quad q = \begin{bmatrix} \frac{11}{20} & \frac{9}{20} \end{bmatrix};$$
that is player R should play the black ace 65% of the time, and player C should play the black 2 55% of the time. The value of the game is $-\frac{3}{20}$ that is, player C can expect to collect on the average 15 cents per game.