1.5

9)
$$A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$$

a) It takes 2 elementary row operations to convert A to I:
1.) multiply Row 1 by 5 and add to Row 2
2) multiply Row 2 by 1/2

If we apply each of these operations to I we get the 2 elementary matrices.

$$E_1 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

From Theorem 1.5.1, we conclude that: $E_2 E_1 A = I$

b) Since the inverse of a matrix is unique, the above equation guarantees that (with E_1 and E_2 as defined above)

$$A^{-1} = E_2 E_1$$

c) From the equation $E_2 E_1 = A$, we conclude that

$$E_2^{-1} E_2 E_1 A = E_2^{-1} I$$

or $\quad E_1 A = E_2^{-1}$

or $\quad E_1^{-1} E_1 A = E_1^{-1} E_2^{-1}$

or $\quad A = E_1^{-1} E_2^{-1}$

so that
$$A = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

15.) The hypothesis that B is row equivalent to A can be expressed as follows:

$$B = E_k E_{k-1} \dots E_1 A$$

Where $E_1, E_2 \dots E_k$ are elementary matrices. Since A is invertible, then (by the rule following Thm. 1.5.3) B is invertible and

$$B^{-1} = A^{-1} E_1^{-1} E_2^{-1} \dots E_k^{-1}$$

1.6

2.)

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -5 \end{bmatrix} \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad b = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad A^{-1} = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ 2 & -4 \end{bmatrix} \rightarrow \text{Thus} \quad X = A^{-1}b$$

$$X = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 9 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -42 \\ -42 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \quad \boxed{X_1 = X_2 = -3}$$

4.)

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \quad X = A^{-1}b$$

Compute A^{-1}

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] =$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 3 & 2 & -3/2 & 5/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \\ 0 & 0 & 1 & 3/2 & -5/2 & 3 \end{array}\right]$$

$$X = A^{-1}b = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix} \quad \boxed{\begin{array}{l} X_1 = 1 \\ X_2 = -11 \\ X_3 = 16 \end{array}}$$

8.)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad x = A^{-1}b$$

Compute A^{-1}

$$A^{-1} = \begin{bmatrix} -15/2 & 1/2 & 5/2 \\ 1/2 & 1/2 & -1/2 \\ 5/2 & -1/2 & -1/2 \end{bmatrix} \qquad \begin{bmatrix} -15/2 & 1/2 & 5/2 \\ 1/2 & 1/2 & -1/2 \\ 5/2 & -1/2 & -1/2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} =$$

$$\boxed{\begin{array}{l} X_1 = -15/2\, b_1 + 1/2\, b_2 + 5/2\, b_3 \\ X_2 = 1/2\, b_1 + 1/2\, b_2 - 1/2\, b_3 \\ X_3 = 5/2\, b_1 - 1/2\, b_2 - 1/2\, b_3 \end{array}}$$

12.) Two systems:

$$-x_1 + 4x_2 + x_3 = 0 \qquad -x_1 + 4x_2 + x_3 = -3$$
$$x_1 + 9x_2 - 2x_3 = 1 \qquad x_1 + 9x_2 - 2x_3 = 4$$
$$6x_1 + 4x_2 - 8x_3 = 0 \qquad 6x_1 + 4x_2 - 8x_3 = -5$$

Augment the coefficient matrix with the columns of constants on the right side.

$$\left[\begin{array}{ccc|c|c} -1 & 4 & 1 & 0 & -3 \\ 1 & 9 & -2 & 1 & 4 \\ 6 & 4 & -8 & 0 & -5 \end{array}\right]$$ Reduce to row-echelon form

$$\left[\begin{array}{ccc|c|c} -1 & 4 & 1 & 0 & -3 \\ 0 & 13 & -1 & 1 & 1 \\ 0 & 28 & -2 & 0 & -23 \end{array}\right] = \left[\begin{array}{ccc|c|c} -1 & 4 & 1 & 0 & -3 \\ 0 & 1 & 0 & -1 & -25/2 \\ 0 & 14 & -1 & 0 & -23/2 \end{array}\right] =$$

$$= \left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 & -25/2 \\ 0 & 0 & 1 & -14 & -327/2 \end{array}\right] = \left[\begin{array}{ccc|c|c} 1 & -4 & 0 & -14 & -321/2 \\ 0 & 1 & 0 & -1 & -25/2 \\ 0 & 0 & 1 & -14 & -327/2 \end{array}\right] =$$

$$= \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -18 & -421/2 \\ 0 & 1 & 0 & -1 & -25/2 \\ 0 & 0 & 1 & -14 & -327/2 \end{array}\right]$$

a) $x_1 = -18 \quad x_2 = -1 \quad x_3 = -14$

b) $x_1 = -421/2 \quad x_2 = -25/2 \quad x_3 = -327/2$

18. $\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{bmatrix}$ Reduce to row-echelon form

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2-b_3 \\ 0 & -1 & 0 & b_3+4b_1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 0 & -1 & b_2-b_3-b_1 \\ 0 & 1 & 0 & -b_3-4b_1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & -1 & b_1-2b_3-8b_1 \\ 0 & 0 & 1 & b_3+b_1-b_2 \\ 0 & 1 & 0 & -b_3-4b_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -6b_1-b_2-b_3 \\ 0 & 1 & 0 & -b_3-4b_1 \\ 0 & 0 & 1 & b_3+b_1-b_2 \end{bmatrix}$$

No restrictions

21. $$\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}}_{A} X = \underbrace{\begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}}_{B}$$

Compute A^{-1} $\quad AX = B \implies A^{-1}AX = A^{-1}B \quad X = A^{-1}B$

$$A^{-1}\cdot B = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & -35 \end{bmatrix}$$