Problem Set Solutions
Section 2.3.

3.) $\det(A)=0$ because, since row 3 = row 1 + row 2. if were to reduce the matrix we would end up with a row of zeros and thus the $\det(A)=0$.

4. a) $A=\begin{bmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{bmatrix}$ Invertible $\det A \neq 0$

b) $B=\begin{bmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{bmatrix}$ Not invertible because $\det B = 0$

c) $C=\begin{bmatrix} \sqrt{2} & -\sqrt{7} & 0 \\ 3\sqrt{2} & -3\sqrt{7} & 0 \\ 5 & -9 & 0 \end{bmatrix}$ Not invertible because $\det C = 0$

d) $D=\begin{bmatrix} -3 & 0 & 1 \\ 5 & 0 & 6 \\ 8 & 0 & 3 \end{bmatrix}$ Not invertible because $\det D = 0$

5. $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\det(A)=-7$

a) $\det(3A) = 3^3 \det(A) = 27 \cdot (-7) = -189$

b) $\det(A^{-1}) = \dfrac{1}{\det A} = -\dfrac{1}{7}$

c) $\det(2A^{-1}) = 2^3 \det(A^{-1}) = 2^3 \frac{1}{\det A} =$

$= 8 \cdot -\frac{1}{7} = -\frac{8}{7}$

d) $\det((2A)^{-1}) \Rightarrow \det(2A) = 2^3 \cdot (-7) = -56$

$\det((2A)^{-1}) = \frac{1}{\det(2A)} = \frac{-1}{56}$

e)

$A = \begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix} = -\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$ Interchange Columns 2 and 3

$= -\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ Take the transpose of the matrix

$\det A = -\det(A) = -(-7) = 7$

g. $\det \begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = \det \begin{vmatrix} 2a_1 & a_1-b_1 & c_1 \\ 2a_2 & a_2-b_2 & c_2 \\ 2a_3 & a_3-b_3 & c_3 \end{vmatrix}$ Add 1, and 2. column.

$= 2 \begin{vmatrix} a_1 & a_1-b_1 & c_1 \\ a_2 & a_2-b_2 & c_2 \\ a_3 & a_3-b_3 & c_3 \end{vmatrix} = 2 \begin{vmatrix} a_1 & -b_1 & c_1 \\ a_2 & -b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{vmatrix}$ Subtract column 1 from column 2

$= -2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

14.) a) $x_1 + 2x_2 = \lambda x_1$
$2x_1 + x_2 = \lambda x_2$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which can be simplified to

$$\begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) $2x_1 + 3x_2 = \lambda x_1$
$4x_1 + 3x_2 = \lambda x_2$

$$\begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which can be simplified to

$$\begin{bmatrix} \lambda - 2 & -3 \\ -4 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

c) $3x_1 + x_2 = \lambda x_1$
$-5x_1 - 3x_2 = \lambda x_2$

$$\begin{bmatrix} 3 & 1 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -5 & -3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 3 & -1 \\ 5 & \lambda + 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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15.)

a) i) characteristic equation

$$\det(\lambda I - A) = \begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{bmatrix} = (\lambda-1)^2 - 4 =$$

$$= \lambda^2 - 2\lambda - 3$$, so char eqn. is $\underline{\lambda^2-2\lambda-3=0}$

ii) eigenvalues

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda-3)(\lambda+1) = 0$$

$$\boxed{\lambda_1 = 3} \quad \boxed{\lambda_2 = -1}$$

iii) eigenvectors

$\lambda_1 = 3$, corresponding eigenvector are the nonzero solutions $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the equation

$$\begin{bmatrix} 3-1 & -2 \\ -2 & 3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2 = t \quad \text{so} \quad x = \begin{bmatrix} t \\ t \end{bmatrix}, t \neq 0$$

$\lambda_2 = -1$

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\underline{x_1 + x_2 = 0} \quad x_1 = -x_2 \quad \underline{x_2 = t \quad x_1 = -t}$

so $x = \begin{bmatrix} -t \\ t \end{bmatrix}$

b) i) characteristic equation

$$\det(\lambda I - A) = \begin{bmatrix} \lambda-2 & -3 \\ -4 & \lambda-3 \end{bmatrix} = (\lambda-2)(\lambda-3) - 12 =$$

$$\lambda^2 - 5\lambda - 6$$

characteristic equation $\underline{\lambda^2 - 5\lambda - 6 = 0}$

ii) eigenvalues

$(\lambda-6)(\lambda+1)=0$

$\lambda_1=6 \quad \lambda_2=-1$

iii) eigenvectors

$\lambda_1=6$

$$\begin{bmatrix} 6-2 & -3 \\ -4 & 6-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution to this system is $x_1=(3/4)t$, $x_2=t$, and eigenvector is $\Rightarrow$ $\underline{x=\begin{bmatrix} (3/4)t \\ t \end{bmatrix}}$, $t\neq 0$

$\lambda_2=-1$

$$\begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1=t \quad x_2=-t$, so the eigenvector is $x=\begin{bmatrix} t \\ -t \end{bmatrix}$, $\underline{t\neq 0}$

c) i) characteristic equation

$$\det(\lambda I-A)=\begin{bmatrix} \lambda-3 & -1 \\ 5 & \lambda+3 \end{bmatrix} = (\lambda-3)(\lambda+3)+5 =$$

$$=\lambda^2-9+5=\lambda^2-4$$

characteristic equation $\quad \lambda^2-4=0$

ii) eigenvalues

$(\lambda-2)(\lambda+2)=0 \qquad \lambda_1=2 \qquad \lambda_2=-2$

iii) eigenvectors

$\lambda=2$

$$\begin{bmatrix} -1 & -1 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_1=-x_2 \quad x_2=t \quad x_1=-t \quad x=\begin{bmatrix} -t \\ t \end{bmatrix}$$

$\underline{t\neq 0}$

$\lambda_2=-2$

$$\begin{bmatrix} -5 & -1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_2=-\frac{1}{5}x_2 \quad x_2=t \quad x_1=-\frac{1}{5}t \quad x=\begin{bmatrix} -t/5 \\ t \end{bmatrix}$$

$t\neq 0$