$\boxed{4.3.1}$

1.) a) ~~Not~~ one-to-one b/c more than one vector can have the same orthogonal projection on the x-axis.

b) One-to-one b/c, reflection is its own reverse, it is a one-to-one mapping of $\mathbb{R}^2$ onto itself.

c) One-to-one because, - same as b)

d) One-to-one b/c the standard matrix for this operation has $\det([T]) \neq 0$

e) One-to-one b/c, same as d)

f) One-to-one, same as b) (except it is $\mathbb{R}^3$)

g) One-to-one, same as d)

5. c.) $w_1 = -x_2$, $w_2 = -x_1$ $\quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ $\quad [T]^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [T^{-1}]$

$T^{-1}(w_1, w_2) = (-w_2, -w_1)$

d) $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$[T] = \begin{bmatrix} 3 & 0 \\ -5 & 0 \end{bmatrix}$ $\quad \det([T]) = 0$

Since $\det([T]) = 0$, T is not one-to-one

6c.) $T = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -7 & 4 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] =$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -7 & 4 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -7 & 4 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & -3/2 & 11/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -3/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & -1/2 \end{array}\right]$$

$[T]^{-1} = \begin{bmatrix} -3/2 & -3/2 & 11/2 \\ 1/2 & 1/2 & -3/2 \\ -1/2 & 1/2 & -1/2 \end{bmatrix}$ $\quad T^{-1}(w_1, w_2, w_3) = \left(\frac{-3w_1 - 3w_2 + 11}{2}, \frac{w_1 + w_2 - 3w_3}{2}, \frac{-w_1 + w_2 - w_3}{2}\right)$

d) $[T] = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 4 \\ 7 & 4 & -5 \end{bmatrix}$ $\quad \det[T] = 0$ $\quad \mathbb{R}^3 \to \mathbb{R}^3$ is ~~not~~ one-to-one

7.) a) Reflection about the x-axis in $\mathbb{R}^2$ $\quad [T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad [T^{-1}] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b) The rotation through an angle of $\pi/4$ in $\mathbb{R}^2$

$[T] = \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix}$ $\quad [T^{-1}] = \begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix}$

c) contraction by a factor of 1/3

$[T] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad [T^{-1}] = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$

d) Reflection about the yz-plane

$[T] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad [T^{-1}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) Dilation by a factor of 5

$[T] = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$ $\quad [T^{-1}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

8. c) $T(x, y) = (-y, x)$

$T((x_1, y_1) + (x_2, y_2)) = (-(y_1 + y_2), x_1 + x_2)$
$= (-y_1, x_1) + (-y_2, x_2)$
$= T(x_1, y_1) + T(x_2, y_2)$

$T[k(x, y)] = T(kx, ky)$
$= (-ky, kx) = k(-y, x) = kT(x, y)$

linear

d) $T(x, y) = (x, 0)$

$T(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 0) =$
$(x_1, 0) + (x_2, 0) =$
$T(x_1, y_1) + T(x_2, y_2)$

$T(k(x, y)] = T(kx, y) = (kx, 0) = k(x, 0) =$
$kT(x, y)$

linear

14. b) $e_1=(1,0,0)\Rightarrow(1,0,0)\Rightarrow(1,0,0)$
$e_2=(0,1,0)\Rightarrow(0,0,0)\Rightarrow(0,0,0)$
$e_3=(0,0,1)\Rightarrow(0,0,1)\Rightarrow(0,0,0)$

$$T=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

c) $e_1=(1,0,0)\Rightarrow(1,0,0)\Rightarrow(1,0,0)\Rightarrow(-1,0,0)$
$e_2=(0,1,0)\Rightarrow(0,1,0)\Rightarrow(0,-1,0)\Rightarrow(0,-1,0)$
$e_3=(0,0,1)\Rightarrow(0,0,-1)\Rightarrow(0,0,-1)\Rightarrow(0,0,-1)$

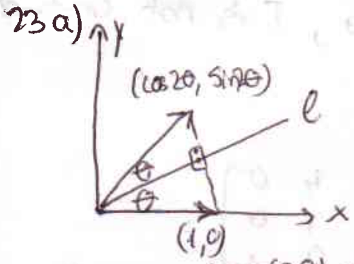
$$T=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

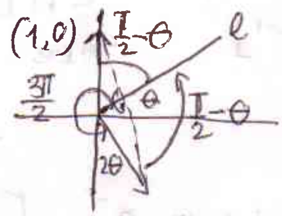
15. a) $T_A(e_1)=\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\4\end{bmatrix}$ $\quad T_A(e_2)=\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}3\\1\\5\end{bmatrix}$ $\quad T_A(e_3)=\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\2\\-3\end{bmatrix}$

b) $T_A(e_1)+T_A(e_2)+T_A(e_3)=\begin{bmatrix}-1\\2\\4\end{bmatrix}+\begin{bmatrix}3\\1\\5\end{bmatrix}+\begin{bmatrix}0\\2\\-3\end{bmatrix}=\begin{bmatrix}2\\5\\0\end{bmatrix}$

c) $T_A(7e_3)=7\begin{bmatrix}0\\2\\-3\end{bmatrix}=\begin{bmatrix}0\\14\\-21\end{bmatrix}$

23 a)





$T(e_2)=(\cos(\frac{3\pi}{2}+2\theta),\sin(\frac{3\pi}{2}+2\theta))$
$=(\sin 2\theta,-\cos 2\theta)$

$T(e_1)=(\cos(2\theta),\sin(2\theta))$

b) $$[T]=\begin{bmatrix}\cos(2(\frac{\pi}{6}))&\sin(2(\frac{\pi}{6}))\\\sin(2(\frac{\pi}{6}))&-\cos(2(\frac{\pi}{6}))\end{bmatrix}=\begin{bmatrix}1/2&\sqrt{3}/2\\\sqrt{3}/2&-1/2\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\5\end{bmatrix}\right)=\begin{bmatrix}1/2&\sqrt{3}/2\\\sqrt{3}/2&-1/2\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}(1+5\sqrt{3})/2\\(\sqrt{3}-5)/2\end{bmatrix}$$

Exercise Set 7.1

7. a) $\det(\lambda I-A)=0$

$$=\det\begin{pmatrix}\lambda&0&-2&0\\-1&\lambda&-1&0\\0&-1&\lambda+2&0\\0&0&0&\lambda-1\end{pmatrix}=0$$

$\lambda^4+\lambda^3-3\lambda^2-\lambda+2=0$
$(\lambda-1)^2(\lambda+2)(\lambda+1)=0$

b) $$\det\begin{pmatrix}\lambda-10&9&0&0\\-4&\lambda+2&0&0\\0&0&\lambda+2&7\\0&0&-1&\lambda-2\end{pmatrix}=0$$

$\lambda^4-8\lambda^3+19\lambda^2-24\lambda+48=0$
$(\lambda-4)^2(\lambda^2+3)=0$

8 a) $\lambda=1,-2,-1$ $\quad$ b) $\lambda=4$ $\quad\lambda^2=-3$ No solution

10. By Theorem 7.1.1 a) $\lambda=-1,\lambda=5$ b) $\lambda=3,\lambda=7,\lambda=1$ c) $\lambda=-1/3,\lambda=1$ $\lambda=\frac{1}{2}$

11. By Theorem 7.1.1. $\lambda=1,\ 1/2,\ 0,\ 2$ Eigenvalues $A^9\Rightarrow 1,\ 1/512,\ 0,\ 512$

13. The vectors Ax and x will lie on the same line if there is a real number λ, such that $Ax = \lambda x$.

a) $\det\begin{pmatrix} \lambda-4 & 1 \\ -2 & \lambda-1 \end{pmatrix} = 0$ $\quad \lambda^2 - 5\lambda + 6 = 0 \quad \lambda = 3 \quad \lambda = 2$

$\lambda = 3 \quad x = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda = 2 \quad x = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\underline{y = x}$ and $\underline{y = 2x}$ are invariant under A

b) $\det\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} = 0 \quad \lambda^2 + 1 = 0$ $\quad$ A has no real eigenvalues, so there are no lines which are invariant under A

c) $\det\begin{pmatrix} \lambda-2 & 3 \\ 0 & \lambda-2 \end{pmatrix} = 0 \quad (\lambda-2)^2 = 0 \quad \lambda = 2 \quad x = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\underline{y = 0}$ is invariant under A

16. let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad \lambda^2 - tr(A)\lambda + \det(A) = 0$

$\det(\lambda I - A) = 0$

$\det\begin{pmatrix} \lambda-a & b \\ c & \lambda-d \end{pmatrix} = 0$

$(\lambda - a)(\lambda - d) - bc = 0$

$\lambda^2 - \lambda d - \lambda a + ad - bc = 0$

$\lambda^2 - \lambda(a+d) + ad - bc = 0$

$\lambda^2 - tr(A)\lambda + \det(A) = 0$

22. $\begin{vmatrix} \lambda+2 & -2 & -3 \\ 2 & \lambda-3 & -2 \\ 4 & -2 & \lambda-5 \end{vmatrix} = 0$ $\qquad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$(\lambda-3)(\lambda-2)(\lambda-1) = 0$

$\lambda = 1, 2, 3$

a) $\lambda = 1, 1/2, 1/3$

b) $A - 3I = \begin{bmatrix} -5 & 2 & 3 \\ -2 & 0 & 2 \\ -4 & 2 & 2 \end{bmatrix}$ $\quad \begin{vmatrix} \lambda+5 & -2 & -3 \\ 2 & \lambda & -2 \\ 4 & -2 & \lambda-2 \end{vmatrix} = 0$ $\quad \lambda^3 + 3\lambda^2 + 2\lambda = 0$

$\lambda(\lambda+1)(\lambda+2) = 0$

$\lambda = 0, -1, -2$

c) $A + 2I = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 5 & 2 \\ -4 & 2 & 7 \end{bmatrix}$ $\quad \begin{vmatrix} \lambda & -2 & -3 \\ 2 & \lambda-5 & -2 \\ 4 & -2 & \lambda-7 \end{vmatrix} = 0$

$\lambda^3 - 12\lambda^2 + 47\lambda - 60 = 0$

$(\lambda-5)(\lambda-4)(\lambda-3) = 0$

$\lambda = 5, 4, 3$