Math 20 Fall 2003 Name SOLUTIONS

Midterm 1

1. Solve the following system of linear equations

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & - & 5x_3 & = & -1 \\ -2x_1 & + & 2x_2 & + & 11x_3 & = & 1 \\ 3x_1 & - & x_2 & + & x_3 & = & 3 \end{array}$$

$$\begin{bmatrix} 1 & -1 & -5 & -1 \\ -2 & 2 & 11 & 1 \\ 3 & -1 & 1 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -5 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & 16 & 6 \end{bmatrix} \quad \begin{array}{l} 2\text{ row \#1} + \text{row \#2} \\ -3\text{ row \#1} + \text{row \#3} \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -5 & -1 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} \tfrac{1}{2}\text{ row \# 3} \\ \text{row \# 2} \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & -6 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} 5\text{ row \# 3} + \text{row \# 1} \\ -8\text{ row \# 3} + \text{row \# 2} \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{row \# 2} + \text{row \# 1}$$

$$\Rightarrow \boxed{\begin{array}{l} x_1 = 5 \\ x_2 = 11 \\ x_3 = -1 \end{array}}$$

2. Given that $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, find $(3AB)^T - B^T A^T$.

$$(3AB)^T - B^T A^T = 3(AB)^T - (AB)^T = 2(AB)^T$$

$$AB = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 3 & 1 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 7 & 3 \\ 9 & 1 \end{bmatrix}$$

$$2(AB)^T = \begin{bmatrix} 14 & 6 \\ 18 & 2 \end{bmatrix}$$

3. Given that $(I - 3A)^{-1} = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$, find the matrix A.

$$I - 3A = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix}$$

$$3A = I - \begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 2 & 0 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} 1/2 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1/6 & 0 \\ 2/3 & 0 \end{bmatrix}$$

4. Given that $A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 2 & 10 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 2 & -3 \end{bmatrix}$, find an elementary matrix E such that $EA = B$.

-2 row#1 + row #3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[+\text{row }\#3]{-2\text{row}\#1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = E$$

5. Given that $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right] \begin{array}{l} \\ \tfrac{1}{2}\text{ row }\#2 \\ \tfrac{1}{4}\text{ row }\#3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -1/4 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right] \begin{array}{l} -\text{row }\#3 + \text{row }\#1 \\ -\text{row }\#3 + \text{row }\#2 \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1/4 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right] \quad -2\text{ row }\#2 + \text{row }\#1$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1/4 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/4 \end{bmatrix}$$

6. Given that $A = \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}$ and $\det(A) = 4$, evaluate the following determinants.

(a) $\begin{vmatrix} 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$ $= (3)(1)(-2)(4) = -24$

(b) $\begin{vmatrix} 1 & -2 & 3 & -4 \\ 0 & 5 & -6 & 7 \\ 0 & 0 & -8 & 9 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ $= (1)(5)(-8)(1) = -40$

(c) $\begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix}$ interchanged row #1 and row #3

$\parallel$

$\det(A) = -4$

(d) $\begin{vmatrix} r & s & t \\ 8u & 8v & 8w \\ x-8r & y-8s & z-8t \end{vmatrix}$ 8 row #2
-8 row #1 + row #3

$\parallel$

$8 \det(A) = 8 \cdot 4 = 32$

(e) $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{4}$

(f) $\det(A^2) = \det(A) \cdot \det(A) = 4 \cdot 4 = 16$

7. Evaluate the determinant $\begin{vmatrix} 1 & 7 & 2 & 0 \\ 2 & 5 & 5 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$.

$$= 2 \begin{vmatrix} 1 & 7 & 2 \\ 2 & 5 & 5 \\ 1 & 1 & -1 \end{vmatrix} = 2\left(1 \begin{vmatrix} 5 & 5 \\ 1 & -1 \end{vmatrix} - 7 \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix}\right)$$

$$= 2\left((-5-5) - 7(-2-5) + 2(2-5)\right)$$

$$= 2(-10 - 7(-7) + 2(-3))$$

$$= 2(-10 + 49 - 6) = 2(33) = 66$$

8. Let $A = \begin{bmatrix} 3 & 0 \\ 1 & 9 \end{bmatrix}$.

(a) Find the eigenvalues of the matrix A.

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 0 \\ -1 & \lambda - 9 \end{vmatrix} = (\lambda - 3)(\lambda - 9)$$

$$\Rightarrow \lambda = 3, 9$$

(b) Select one of the eigenvalues you found in part (a) and find its corresponding eigenvector.

$\lambda = 3$: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & -6 & 0 \end{array}\right] \Rightarrow -x_1 - 6x_2 = 0 \Rightarrow x_1 = -6x_2$

$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6x_2 \\ x_2 \end{bmatrix} = s\begin{bmatrix} -6 \\ 1 \end{bmatrix}$

$\lambda = 9$: $\left[\begin{array}{cc|c} 6 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 6 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow 6x_1 = 0 \Rightarrow x_1 = 0$

$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

9. Explain why the following statement is true. "If $\det(A) = 0$, then the homogeneous system $Ax = 0$ has infinitely many solutions."

Every homogeneous system has either just the trivial solution or infinitely many solutions. If $Ax=0$ has only the trivial solution, then $\det(A) \neq 0$ by the Big Theorem. Since $\det(A) = 0$, it must be the case that $Ax = 0$ has more than the trivial solution — in which case it must have infinitely many solutions.

10. In each round of the two-person game ***rock-paper-scissors***, each player chooses one of rock, paper, or scissors. Rock beats scissors, scissors beats paper, and paper beats rock. If both players choose the same object, then the round is considered a tie.

(a) Give the payoff matrix for this game, assuming that an entry of 1 indicates a win for the row player, an entry of -1 indicates a loss for the row player, and an entry of 0 indicates a tie. Label the rows and columns of the matrix with the row and column player actions.

$$\begin{array}{c c} & \begin{array}{ccc} \text{Rock} & \text{Paper} & \text{Scissors} \end{array} \\ \begin{array}{c} \text{Rock} \\ \text{Paper} \\ \text{Scissors} \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

(b) If the row player chooses paper 50% of the time and scissors 50% of the time and if the column player chooses rock 75% of the time and scissors 25% of the time, which player will win more rounds of the game on average?

$$p = \begin{bmatrix} 0 & 1/2 & 1/2 \end{bmatrix} \qquad q = \begin{bmatrix} 3/4 \\ 0 \\ 1/4 \end{bmatrix}$$

$$E(p,q) = \begin{bmatrix} 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3/4 \\ 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 3/4 \\ 0 \\ 1/4 \end{bmatrix}$$

$$= -1/8 < 0 \Rightarrow \text{column player}$$