

Midterm 2

1. Find an equation for the plane containing the line

$$\begin{aligned}x &= 3 + 6t \\y &= 4 \\z &= t\end{aligned}$$

and that is parallel to the line of intersection of the planes $2x + y + z = 1$ and $x - 2y + 3z = 2$.

line of intersection has direction vector

$$(2, 1, 1) \times (1, -2, 3) = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = (5, 5, -5)$$

normal vector to target plane is \perp to this vector
and to direction vector of line:

$$(5, 5, -5) \times (6, 0, 1) = \begin{vmatrix} i & j & k \\ 5 & 5 & -5 \\ 6 & 0 & 1 \end{vmatrix} = (-5, -35, 30)$$

any point on the line is a point in the plane:

$$t = 0 \Rightarrow (3, 4, 0)$$

point-normal form:

$$-5(x-3) - 35(y-4) + 30z = 0$$

$$\text{or } -x - 7y + 6z = -31$$

2. Which of the following are true for any vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in \mathbb{R}^3 and any scalar k ?

(a) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

TRUE

(b) If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(c) $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$

(d) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

TRUE

(e) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

TRUE

3. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the one-to-one linear operator defined by the following equations. Find $T^{-1}(2, 3)$.

$$\begin{aligned}w_1 &= 2x_1 - x_2 \\w_2 &= 8x_1 - 5x_2\end{aligned}$$

$$[T] = \begin{bmatrix} 2 & -1 \\ 8 & -5 \end{bmatrix}$$

$$[T^{-1}] = [T]^{-1} = \frac{-1}{2} \begin{bmatrix} -5 & 1 \\ -8 & 2 \end{bmatrix} = \begin{bmatrix} 5/2 & -1/2 \\ 4 & -1 \end{bmatrix}$$

$$T^{-1}(2, 3) = \begin{bmatrix} 5/2 & -1/2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 5 \end{bmatrix}$$

$$(7/2, 5)$$

4. Determine the standard matrix for the linear operator on \mathbb{R}^3 which rotates by $\frac{\pi}{4}$ radians about the z -axis and then reflects about the yz -plane.

$$\theta = \frac{\pi}{4}$$
$$[T_1] = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T] = [T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Find the eigenvalues of A^4 given that $A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}$.

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 5 & 3 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda - 5)(\lambda - 1) + 3$$

$$= \lambda^2 - 6\lambda + 5 + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$$

A has eigenvalues 2 and 4

$\Rightarrow A^4$ has eigenvalues 2^4 and 4^4

16 and 256

6. Determine whether the following matrix is diagonalizable. Explain your reasoning.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda - 3 & -2 \\ 0 & 1 & \lambda \end{vmatrix} = \lambda \left((\lambda - 3)\lambda + 2 \right)$$

$$= \lambda (\lambda^2 - 3\lambda + 2) = \lambda (\lambda - 1)(\lambda - 2)$$

Since A is a 3×3 matrix with 3 distinct eigenvalues $(0, 1, \text{ and } 2)$, it follows that A has 3 linearly independent eigenvectors.

This implies that A is diagonalizable.

7. Suppose that A is an $n \times n$ matrix and $T_A : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is multiplication by A . Explain why the following statement is true. "If $\det(\lambda I - A) = \lambda^n + 1$, then there is no nonzero vector x in \mathbf{R}^n such that $T_A(x) = 0$."

Since $\det(0 \cdot I - A) = 0^n + 1 = 1 \neq 0$, it follows that 0 is not an eigenvalue of A . This implies by the Big Theorem that $Ax = 0$ has only the trivial solution. Since $T_A(x) = Ax$, this implies that the only solution to $T_A(x) = 0$ is the zero vector. Hence there is no nonzero vector x such that $T_A(x) = 0$.

8. Suppose that a group of three people, Larry, Moe, and Curly, work together on a farm. All of the food produced by these three people is consumed among the three. In addition, the amount of food produced by each person equals the total amount consumed by that person. (For example, Larry eats the same amount of food that he produces.)

Of the food produced by Larry, $\frac{1}{3}$ is consumed by Larry himself and $\frac{2}{3}$ is consumed by Moe. Of the food produced by Moe, $\frac{1}{2}$ is consumed by Larry and $\frac{1}{2}$ is consumed by Curly. Of the food produced by Curly, $\frac{1}{4}$ is consumed by Moe and $\frac{3}{4}$ is consumed by Curly himself.

Assuming that Curly consumes 32 pounds of food a week, how much food do Larry and Moe consume a week?

$$\begin{array}{c}
 L \\
 M \\
 C
 \end{array}
 \begin{array}{c}
 L \quad M \quad C \\
 \left[\begin{array}{ccc}
 1/3 & 1/2 & 0 \\
 2/3 & 0 & 1/4 \\
 0 & 1/2 & 3/4
 \end{array} \right]
 \end{array}$$

$x_1 =$ lbs. consumed by Larry

$x_2 =$ " " " Moe

$x_3 =$ " " " Curly

$= E$

want $Ex = x \Leftrightarrow (I - E)x = 0$

$$\left[\begin{array}{ccc|c}
 2/3 & -1/2 & 0 & 0 \\
 -2/3 & 1 & -1/4 & 0 \\
 0 & -1/2 & 1/4 & 0
 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c}
 1 & -3/4 & 0 & 0 \\
 0 & 1/2 & -1/4 & 0 \\
 0 & -1/2 & 1/4 & 0
 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c}
 1 & -3/4 & 0 & 0 \\
 0 & 1 & -1/2 & 0 \\
 0 & 0 & 0 & 0
 \end{array} \right]$$

$$\Rightarrow x_1 = 3/4 x_2 \text{ and } x_2 = 1/2 x_3 \Rightarrow x_1 = \frac{3}{4} \left(\frac{1}{2} x_3 \right) = \frac{3}{8} x_3$$

Since $x_3 = 32$, we have

$$\begin{array}{l}
 x_2 = 16 \leftarrow \text{Moe} \\
 x_1 = 12 \leftarrow \text{Larry}
 \end{array}$$