2. $\begin{bmatrix} 1 & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 7 & 0 \\ 0 & \textcircled{-5} & 10 & 0 \\ 0 & 0 & \textcircled{12} & 0 \end{bmatrix}$

There is no free variable; the system has only the trivial solution.

6. $\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 4 & 0 \\ 0 & \textcircled{1} & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned} \textcircled{x_1} \quad + 4x_3 &= 0 \\ \textcircled{x_2} - 3x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

The variable x_3 is free, $x_1 = -4x_3$, and $x_2 = 3x_3$.

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.

12. $\begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 5 & 0 & 8 & 1 & 0 & 0 \\ 0 & 0 & \textcircled{1} & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned} \textcircled{x_1} + 5x_2 \quad + 8x_4 + x_5 \quad &= 0 \\ \textcircled{x_3} - 7x_4 + 4x_5 \quad &= 0 \\ \textcircled{x_6} &= 0 \\ 0 &= 0 \end{aligned}$$

The basic variables are x_1, x_3, and x_6; the free variables are x_2, x_4, and x_5. The general solution is $x_1 = -5x_2 - 8x_4 - x_5$, $x_3 = 7x_4 - 4x_5$, and $x_6 = 0$. In parametric vector form, the solution is

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}-5x_2-8x_4-x_5\\x_2\\7x_4-4x_5\\x_4\\x_5\\0\end{bmatrix}=\begin{bmatrix}-5x_2\\x_2\\0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}-8x_4\\0\\7x_4\\x_4\\0\\0\end{bmatrix}+\begin{bmatrix}-x_5\\0\\-4x_5\\0\\x_5\\0\end{bmatrix}=x_2\begin{bmatrix}-5\\1\\0\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-8\\0\\7\\1\\0\\0\end{bmatrix}+x_5\begin{bmatrix}-1\\0\\-4\\0\\1\\0\end{bmatrix}$$

14. To write the general solution in parametric vector form, pull out the constant terms that do not involve the free variable:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_4 \\ 8+x_4 \\ 2-5x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_4 \\ x_4 \\ -5x_4 \\ x_4 \end{bmatrix} = \underset{\substack{\uparrow \\ \mathbf{p}}}{\begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix}} + x_4 \underset{\substack{\uparrow \\ \mathbf{q}}}{\begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix}} = \mathbf{p} + x_4\mathbf{q}$$

The solution set is the line through **p** in the direction of **q**.

20. The line through **a** parallel to **b** can be written as **x** = **a** + *t***b**, where *t* represents a parameter:

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3\\-4\end{bmatrix}+t\begin{bmatrix}-7\\8\end{bmatrix}, \text{ or } \begin{cases}x_1=3-7t\\x_2=-4+8t\end{cases}$$

26. (*Geometric argument using Theorem* 6.) Since $A\mathbf{x} = \mathbf{b}$ is consistent, its solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, by Theorem 6. So the solution set of $A\mathbf{x} = \mathbf{b}$ is a single vector if and only if the solution set of $A\mathbf{x} = \mathbf{0}$ is a single vector, and that happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*Proof using free variables.*) If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of A is a pivot column. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

30. **a.** When A is a 3×3 matrix with two pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has two basic variables and one free variable. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

b. With only two pivot positions, A cannot have a pivot in every row, so by Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbf{R}^3$).

34. Inspect how the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are related. The second column is –3/2 times the first. Put another way, $3\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{0}$. Thus $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ satisfies $A\mathbf{x} = \mathbf{0}$.

Note: Exercises 33 and 34 set the stage for the concept of linear dependence.