2. Take some other value for p_S, say 200 million dollars. The other equilibrium prices are then $p_C = 188$ million, $p_E = 170$ million. Any constant nonnegative multiple of these prices is a set of equilibrium prices, because the solution set of the system of equations consists of all multiples of one vector. Changing the unit of measurement to, say, European euros has the same effect as multiplying all equilibrium prices by a constant. The *ratios* of the prices remain the same, no matter what currency is used.

4. a. Fill in the exchange table one column at a time. The entries in each column must sum to 1.

	Distribution of Output From:					
	Agric.	Energy	Manuf.	Transp.		Purchased by:
output	↓	↓	↓	↓	input	
	.65	.30	.30	.20	→	Agric.
	.10	.10	.15	.10	→	Energy
	.25	.35	.15	.30	→	Manuf.
	0	.25	.40	.40	→	Transp.

b. Denote the total annual output of the sectors by p_A, p_E, p_M, and p_T, respectively. From the first row of the table, the total input to Agriculture is $.65p_A + .30p_E + .30p_M + .20\,p_T$. So the equilibrium prices must satisfy

income expenses

$$p_A \quad = \; .65\,p_A + .30\,p_E + .30\,p_M + .20\,p_T$$

From the second, third, and fourth rows of the table, the equilibrium equations are

$$\begin{aligned} p_E &= .10\,p_A + .10\,p_E + .15\,p_M + .10\,p_T \\ p_M &= .25\,p_A + .35\,p_E + .15\,p_M + .30\,p_T \\ p_T &= .25\,p_E + .40\,p_M + .40\,p_T \end{aligned}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .35\,p_A - .30\,p_E - .30\,p_M - .20\,p_T &= 0 \\ -.10\,p_A + .90\,p_E - .15\,p_M - .10\,p_T &= 0 \\ -.25\,p_A - .35\,p_E + .85\,p_M - .30\,p_T &= 0 \\ -.25\,p_E - .40\,p_M + .60\,p_T &= 0 \end{aligned}$$

Use gauss, bgauss, and scale operations to reduce the augmented matrix to reduced echelon form

$$\begin{bmatrix} .35 & -.3 & -.3 & -.2 & 0 \\ 0 & .81 & -.24 & -.16 & 0 \\ 0 & 0 & 1.0 & -1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} .35 & -.3 & 0 & -.55 & 0 \\ 0 & .81 & 0 & -.43 & 0 \\ 0 & 0 & 1 & -1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{.35} & 0 & 0 & -.71 & 0 \\ 0 & \textcircled{1} & 0 & -.53 & 0 \\ 0 & 0 & \textcircled{1} & -1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Scale the first row and solve for the basic variables in terms of the free variable p_T, and obtain $p_A = 2.03p_T$, $p_E = .53p_T$, and $p_M = 1.17p_T$. The data probably justifies at most two significant figures, so take $p_T = 100$ and round off the other prices to $p_A = 200$, $p_E = 53$, and $p_M = 120$.

$\to H_3BO_3 + 3H_2S$

6. The following vectors list the numbers of atoms of sodium (Na), phosphorus (P), oxygen (O), barium (Ba), and nitrogen(N):

$$Na_3PO_4: \begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \quad Ba(NO_3)_2: \begin{bmatrix} 0 \\ 0 \\ 6 \\ 1 \\ 2 \end{bmatrix}, \quad Ba_3(PO_4)_2: \begin{bmatrix} 0 \\ 2 \\ 8 \\ 3 \\ 0 \end{bmatrix}, \quad NaNO_3: \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \text{sodium} \\ \text{phosphorus} \\ \text{oxygen} \\ \text{barium} \\ \text{nitrogen} \end{matrix}$$

The coefficients in the equation $x_1 \cdot Na_3PO_4 + x_2 \cdot Ba(NO_3)_2 \rightarrow x_3 \cdot Ba_3(PO_4)_2 + x_4 \cdot NaNO_3$ satisfy

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 6 \\ 1 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 8 \\ 3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Move the right terms to the left side (changing the sign of each entry in the third and fourth vectors) and row reduce the augmented matrix of the homogeneous system:

$$\begin{bmatrix} 3 & 0 & 0 & -1 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 4 & 6 & -8 & -3 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 6 & -8 & -3 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 6 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 6 & 0 & -3 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{bmatrix}$$

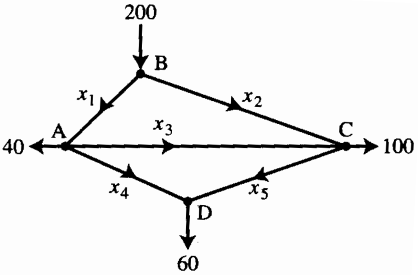
$$\sim\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 18 & -3 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 0 & 6 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $x_1 = (1/3)x_4$, $x_2 = (1/2)x_4$, $x_3 = (1/6)x_4$, with x_4 free. Take $x_4 = 6$. Then $x_1 = 2$, $x_2 = 3$, and $x_3 = 1$. The balanced equation is

$$2Na_3PO_4 + 3Ba(NO_3)_2 \rightarrow Ba_3(PO_4)_2 + 6NaNO_3$$

2. Write the equations for each intersection:

Intersection	Flow in		Flow out
A	x_1	=	$x_3 + x_4 + 40$
B	200	=	$x_1 + x_2$
C	$x_2 + x_3$	=	$x_5 + 100$
D	$x_4 + x_5$	=	60
Total flow:	200	=	200



Rearrange the equations:

$$\begin{array}{rcrcrcrcrcr} x_1 & & & - & x_3 & - & x_4 & & & = & 40 \\ x_1 & + & x_2 & & & & & & & = & 200 \\ & & x_2 & + & x_3 & & & - & x_5 & = & 100 \\ & & & & & & x_4 & + & x_5 & = & 60 \end{array}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -1 & 0 & 1 & 100 \\ 0 & \textcircled{1} & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution (written in the style of Section 1.2) is

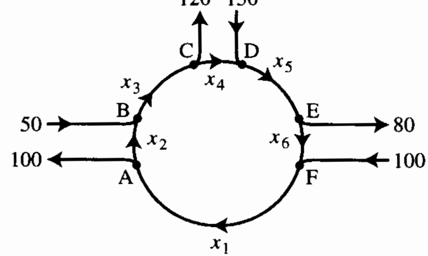
$$\begin{cases} x_1 = 100 + x_3 - x_5 \\ x_2 = 100 - x_3 + x_5 \\ x_3 \text{ is free} \\ x_4 = 60 - x_5 \\ x_5 \text{ is free} \end{cases}$$

b. When $x_4 = 0$, x_5 must be 60, and $\begin{cases} x_1 = 40 + x_3 \\ x_2 = 160 - x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \\ x_5 = 60 \end{cases}$

c. The minimum value of x_1 is 40 cars/minute, because x_3 cannot be negative.

4. Write the equations for each intersection.

Intersection	Flow in		Flow out
A	x_1	$=$	$x_2 + 100$
B	$x_2 + 50$	$=$	x_3
C	x_3	$=$	$x_4 + 120$
D	$x_4 + 150$	$=$	x_5
E	x_5	$=$	$x_6 + 80$
F	$x_6 + 100$	$=$	x_1



Rearrange the equations:

$$
\begin{array}{rcrcrcrcrcrcrcr}
x_1 & - & x_2 & & & & & & & & & & & = & 100 \\
 & & x_2 & - & x_3 & & & & & & & & & = & -50 \\
 & & & & x_3 & - & x_4 & & & & & & & = & 120 \\
 & & & & & & x_4 & - & x_5 & & & & & = & -150 \\
 & & & & & & & & x_5 & - & x_6 & & & = & 80 \\
-x_1 & & & & & & & & & + & x_6 & & & = & -100
\end{array}
$$

Reduce the augmented matrix:

$$
\left[\begin{array}{rrrrrrr}
1 & -1 & 0 & 0 & 0 & 0 & 100 \\
0 & 1 & -1 & 0 & 0 & 0 & -50 \\
0 & 0 & 1 & -1 & 0 & 0 & 120 \\
0 & 0 & 0 & 1 & -1 & 0 & -150 \\
0 & 0 & 0 & 0 & 1 & -1 & 80 \\
-1 & 0 & 0 & 0 & 0 & 1 & -100
\end{array}\right]
\sim \cdots \sim
\left[\begin{array}{rrrrrrr}
1 & -1 & 0 & 0 & 0 & 0 & 100 \\
0 & 1 & -1 & 0 & 0 & 0 & -50 \\
0 & 0 & 1 & -1 & 0 & 0 & 120 \\
0 & 0 & 0 & 1 & -1 & 0 & -150 \\
0 & 0 & 0 & 0 & 1 & -1 & 80 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$\sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ The general solution is } \begin{cases} x_1 = 100 + x_6 \\ x_2 = x_6 \\ x_3 = 50 + x_6 \\ x_4 = -70 + x_6 \\ x_5 = 80 + x_6 \\ x_6 \text{ is free} \end{cases}.$$

Since x_4 cannot be negative, the minimum value of x_6 is 70.