Name: ______________________ ID#: ______________

Solutions to Midterm I

Math 20
Introduction to Linear Algebra
and Multivariable Calculus

25 October 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> *—Handbook for Students*

1. (10 Points) *Let*

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Calculuate

(i) $3A - 2B$

Solution.

$$\begin{aligned} 3A - 2B &= 3\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 3 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ 7 & 6 \end{bmatrix} \end{aligned}$$

□

(ii) $A^T B$

Solution.

$$A^T B = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$$

□

______ / 10

2. (15 Points) *Determine if the matrix*

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 0 \\ -2 & -2 & 3 \end{bmatrix}$$

is invertible and if so, find its inverse.

In this and in any problem where you do Gaussian Elimination, label your row operations to receive full or partial credit.

Solution. We do Gaussian elimination on the matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ -2 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \longmapsto \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 3 & -1 & 1 & 0 \\ 0 & 2 & -3 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 + 2R_1 \end{array}$$

$$\longmapsto \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 3 & -1 & 1 & 0 \\ 0 & 0 & -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 + \frac{1}{2}R_2 \end{array}$$

$$\longmapsto \left[\begin{array}{rrr|rrr} 1 & 2 & 0 & -2 & 1 & -2 \\ 0 & -4 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & -1 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right] \begin{array}{r} R_1 - 2R_3 \\ R_2 + 2R_3 \\ -\frac{2}{3}R_3 \end{array}$$

$$\longmapsto \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right] \begin{array}{r} R_1 + \frac{1}{2}R_2 \\ -\frac{1}{4}R_2 \\ \\ \end{array}$$

So A is invertible, and

$$A^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -1 & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

□

______ / 15

3. (10 Points) *Let* $S = \{\mathbf{v}_1, \dots \mathbf{v}_r\}$ *and let* $V = \operatorname{Span} S$. *Suppose* $\mathbf{u}, \mathbf{w} \in V$. *Show that* $\mathbf{u} + \mathbf{w} \in V$.

Solution. Linear algebra proofs, and proofs in general, are much stricter than they appear. We must use the *definition* of the span, which is this: a vector $\mathbf{a} \in \operatorname{Span} S$ if and only if $\mathbf{a}$ can be written as a linear combination of the elements of S (the span of a set is the set of all linear combinations of elements of that set). What we are trying to show that if $\mathbf{u}$ and $\mathbf{w}$ are both linear combinations of $\{\mathbf{v}_1, \dots \mathbf{v}_r\}$, then so is their sum. Since $\mathbf{u}, \mathbf{w} \in \operatorname{Span} S$, there must be scalars $\{\alpha_1, \dots, \alpha_r\}$ and $\{\beta_1, \dots, \beta_r\}$ such that

$$\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r$$
$$\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_r\mathbf{v}_r$$

So

$$\mathbf{u} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{v}_1 + (\alpha_2 + \beta_2)\mathbf{v}_2 + \cdots + (\alpha_r + \beta_r)\mathbf{v}_r,$$

which is evidently in the span of S.

Another way to do this is to let

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_r \end{bmatrix}.$$

Then to say that $\mathbf{u} \in V$ is equivalent to saying there is a vector $\mathbf{x} \in \mathbb{R}^r$ such that $A\mathbf{x} = \mathbf{u}$. Likewise there is a vector $\mathbf{y} \in \mathbb{R}^r$ such that $A\mathbf{y} = \mathbf{w}$. Then $A(\mathbf{x} + \mathbf{y}) = \mathbf{u} + \mathbf{w}$, which is therefore in V. □

______ / 10

4. (20 Points) *A company is producing high-end fish food out of two ingredients, call them A and B, hoping to minimize costs while still ensuring nutrition for the fish. Each 10g sample of ingredient A costs \$2 and has 2g of nutrient N_1, 4g of nutrient N_2, and 1g of nutrient N_3 (the rest is other nutrients). Each 10g of ingredient B costs \$1 and has 3g nutrient N_1, 1g nutrient N_2, and 1g nutrient N_3. The American Society of Fish Owners advises that fish get at least 12g nutrient N_1, 8g nutrient N_2, and 5g nutrient N_3 per day. What combination of ingredients should be put into each one-day serving of fish food?*

(a) *Formulate the problem as a Linear Programming problem. What are the objective function and constraints?*

Solution. A mixture of x_1 10g-samples of A and x_2 10g-samples of B costs $2x_1 + x_2$ dollars and contains $2x_1 + 3x_2$ grams of N_1, $4x_1 + x_2$ grams of N_2, and $x_1 + x_2$ grams of N_3. So the problem is to minimize $2x_1 + x_2$ subject to the constraints

$$\begin{aligned} 2x_1 + 3x_2 &\geq 12 \\ 4x_1 + x_2 &\geq 8 \\ x_1 + x_2 &\geq 5 \end{aligned}$$

and the usual constraints $x_1, x_2 \geq 0$. □

(b) *Draw the feasibility set.*

Solution. The line $2x_1 + 3x_2 = 12$ has x_1-intercept 6 and x_2-intercept 4. The line $4x_1 + x_2 = 8$ has x_1-intercept 2 and x_2-intercept 8. The line $x_1 + x_2 = 5$ has x_1-intercept 5 and x_2-intercept 5. We find the intersections and plot the region (notice the region is unbounded).

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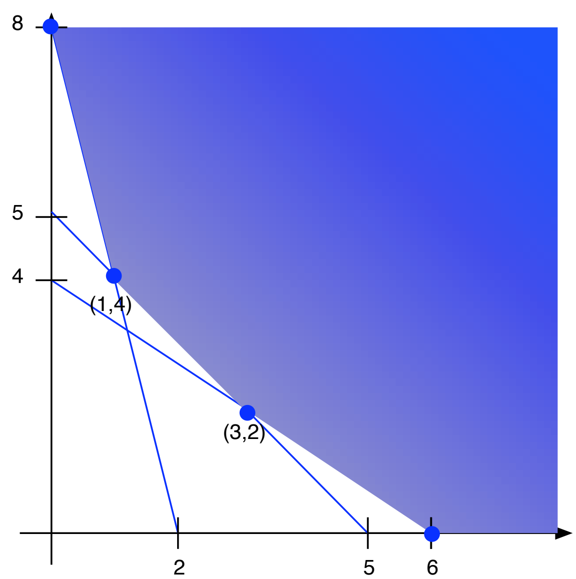

□

(c) Use the Corner Principle to find the solution.

Solution. We need only evaluate the objective function at the four corners.

Combination	Cost
$(0, 8)$	\$8
$(1, 4)$	\$6
$(3, 2)$	\$8
$(6, 0)$	\$12

So the minimum cost is when the $A : B$ ratio is $1 : 4$. The company should combine 10g A and 40g B. □

______ / 20

5. (10 Points) *Let a game have payoff matrix*

$$\begin{bmatrix} 2 & 5 \\ -5 & 0 \end{bmatrix}$$

What are the optimal strategies for the row and column players?

Solution. There is a saddle point in the $(1,1)$ position. Both players should choose the first of the available strategies.

The monstrous formulas for computing the optimal strategies in a 2×2 game do not apply if the game is strictly determined (i.e., when a saddle point exists). If you follow them through, you get "strategies"

$$\mathbf{p}^* = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \end{bmatrix} \qquad \mathbf{q}^* = \begin{bmatrix} -\frac{5}{2} \\ \frac{7}{2} \end{bmatrix} \tag{1}$$

This means player R should make the first choice 250% of the time and the second choice -150% of the time, which is rather difficult. □

______ / 10

6. (15 Points) *A department is trying to schedule teaching fellows for a course. The teaching fellows are polled and have given their preferences of section time. TF A prefers teaching at 9*AM*, 10*AM*, 11*AM*, 12*PM *in that order. TF B prefers teaching at 10, 12, 11, and 9. TF C prefers 10, 9, 12, and 11.*

Find an assignment of teaching fellows to section times that maximizes satisfaction. (Note: you will have to invent a dummy TF who will take the leftover time slot.)

Solution. The Hungarian method is for minimizing costs. If we make the "cost" of an assignment that TF's preference rank, then maximum satisfaction equals minimum total cost. We invent a dummy (so to speak) TF who's completely indifferent. We can make a cost matrix that looks like this:

	A	B	C	D
9 : 00	1	4	2	0
10 : 00	2	1	1	0
11 : 00	3	3	4	0
12 : 00	4	2	3	0

The first step is to subtract the minimum entry from every row, but this is already done since the minimum entry is zero. Then we do the same with the colums, subtracting 1 from each of the first three columns to normalize them (notice that if we had chosen a different value for each of TF D's preferences, this step would zero out that column anyway).

	A	B	C	D
9 : 00	~~0~~	~~3~~	~~1~~	~~0~~
10 : 00	~~1~~	~~0~~	~~0~~	~~0~~
11 : 00	2	2	2	~~0~~
12 : 00	3	(1)	2	~~0~~

As you can see, we can cover all the zeros with only 3 lines, so we need to make further adjustments to get an assignment of zeros. The minimum entry is 1; we subtract that from all uncovered entries and add it to all double-covered entries.

	A	B	C	D
9 : 00	(0)	3	1	1
10 : 00	1	0	(0)	1
11 : 00	1	1	1	(0)
12 : 00	2	(0)	1	0

After this adjustment, we have evidently found an assignment of zeros. This corresponds to giving the 9:00 section to TF A, the noon section to TF B, and the 10:00 section to TF C. TF D (nobody) will teach the 11:00 section. □

______ / 15

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