

Name: \_\_\_\_\_ ID#: \_\_\_\_\_

## Solutions to Midterm II

Math 20  
Introduction to Linear Algebra  
and Multivariable Calculus

22 November 2004

**Show all of your work. Full credit may not be given for an answer alone.** You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

**This is a non-calculator exam.**

*Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*

*—Handbook for Students*

1. (12 Points) Suppose that

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = 7.$$

Find the following, with justification.

(i)

$$\det \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & k \end{bmatrix}$$

*Solution.* The determinant is *multilinear* in each of its rows, so the answer is  $3 \cdot 7 = 21$ .  $\square$

(ii)

$$\det \left( 3 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \right)$$

*Solution.* In this case the effect is that of multiplying *each row* by 3, so the determinant gets multiplied by 3 three times. Thus the determinant of the given matrix is  $3^3 \cdot 7 = 27 \cdot 7 = 189$ .  $\square$

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(iii)

$$\det \begin{bmatrix} d & e & f \\ g & h & k \\ a & b & c \end{bmatrix}$$

*Solution.* Each transposition of the rows introduces a factor of  $-1$ . We have

$$\det \begin{bmatrix} d & e & f \\ g & h & k \\ a & b & c \end{bmatrix} = -\det \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & k \end{bmatrix} = \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = 7.$$

□

(iv)

$$\det \begin{bmatrix} a & b & c \\ d+g & e+h & f+k \\ g & h & k \end{bmatrix}$$

*Solution.* This matrix is our original matrix with the third row added to the second. The determinant does not change under this kind of row operation, and so remains 7. □

2. (8 Points) Suppose  $B$  is a square matrix with  $B^2 = B$ . What can you say about  $\det(B)$ ?

*Solution.* Since the determinant of a product is the product of the determinants, we have  $\det(B)^2 = \det(B)$ , so  $\det(B)$  is either 1 or 0.

Many people decided that  $B^2 = B$  is only true if  $B$  is the identity matrix. There are plenty of  $2 \times 2$  matrices which satisfy this equation (such matrices are called *idempotents*) but aren't the identity. Examples are

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

The idea that  $B^2 = B \implies B = I$  isn't even true for  $1 \times 1$  matrices—i.e., for scalars. The equation  $b^2 = b$  has two solutions:  $b = 1$  and  $b = 0$ .

□

3. (20 Points) Let

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix}.$$

Find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $A = PDP^{-1}$ .

*Hint.* The determinant of this matrix is zero, so this means that 0 is one of the eigenvalues of  $A$ . So the characteristic polynomial won't have a constant term.

*Reminder.* In this and all problems in which you use Gaussian Elimination (row reduction), label each operation to receive partial credit.

*Solution.* The eigenvalues are the roots of the characteristic polynomial:

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 3 - \lambda & 1 & -1 \\ 2 & 2 - \lambda & -2 \\ 1 & -1 & 1 - \lambda \end{vmatrix} \\ &= (3 - \lambda) \begin{vmatrix} 2 - \lambda & -2 \\ -1 & 1 - \lambda \end{vmatrix} - \begin{vmatrix} 2 & -2 \\ 1 & 1 - \lambda \end{vmatrix} - \begin{vmatrix} 2 & 2 - \lambda \\ 1 & -1 \end{vmatrix} \\ &= (3 - \lambda) [(2 - \lambda)(1 - \lambda) - 2] - [2(1 - \lambda) + 2] - [-2 - (2 - \lambda)] \\ &= -\lambda^3 + 6\lambda^2 - 8\lambda \\ &= -\lambda(\lambda - 4)(\lambda - 2). \end{aligned}$$

So the eigenvalues are 0, 2, and 4.

Many tried to replace  $A$  by a row echelon form of  $A$ . This doesn't work: matrices which are row equivalent need not have the same eigenvalues. Try it with

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The matrix on the left-hand side has the eigenvalue 1 with multiplicity 2, but the one on the right-hand side has 1 and  $-1$  as eigenvalues.

To find the eigenvectors, we look at the vectors which generate the solution set to  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  for each  $\lambda$ . We have

$$A - 0I = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A vector spanning this eigenspace has  $x_3$  free,  $x_2 = x_3$ , and  $x_1 = 0$ . So

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

is a good choice.

### 3

### 3

For the second eigenvalue,

$$A - 2I = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So  $x_3$  is free,  $x_2 = 0$ , and  $x_1 = x_3$ . Thus

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector.

Finally,

$$A - 4I = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -2 & -2 \\ 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So  $x_3 = 0$ ,  $x_2$  is free, and  $x_1 = x_2$ . Thus

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

is an eigenvector for this eigenspace.

Keep in mind that if the system of linear equations  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  has only the trivial solution, then  $\lambda$  is not an eigenvalue!

Putting this all together, we have

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

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□

4. (13 Points) Suppose that the probability that the child of a college-educated parent also becomes college-educated is 0.75, and that the probability that the child of a non-college-educated parent becomes college-educated is 0.35.

(a) (5 points) Assuming that the information above describes a Markov chain, write a transition matrix for this situation.

*Solution.* The matrix must be stochastic, so we have all the information we need:

$$P = \begin{bmatrix} 0.75 & 0.35 \\ 0.25 & 0.65 \end{bmatrix}.$$

□

(b) (3 points) If 30% of parents are college-educated, what proportion of the population will be college-educated next generation?

*Solution.*

$$\begin{bmatrix} 0.75 & 0.35 \\ 0.25 & 0.65 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.47 \\ 0.53 \end{bmatrix}$$

□

(c) (5 points) Without any knowledge of the present proportion of college-educated parents, determine the eventual proportion of college-educated people.

*Solution.* We are looking for the steady-state probability vector, which is an eigenvector of eigenvalue 1. We can find it by row reducing:

$$\begin{aligned} A - I &= \begin{bmatrix} -\frac{1}{4} & \frac{7}{20} \\ \frac{1}{4} & \frac{7}{20} \end{bmatrix} \sim \begin{bmatrix} -\frac{1}{4} & \frac{7}{20} \\ 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -\frac{7}{5} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

So  $x_1 + x_2 = 1$  and  $x_1 = \frac{7}{5}x_2$ . We get  $x_2 = \frac{5}{12}$ , and

$$x_1 = \frac{7}{12},$$

the eventual proportion of college-educated people.  $\square$

5. (12 Points) Consider the set of vectors in  $\mathbb{R}^4$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

Find an orthonormal set of vectors  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  with the same span. (Use the Gram-Schmidt orthogonalization process.)

*Solution.* This is a simple evaluation. First, we orthogonalize:

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \\ \mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \\ &= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3} \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \\ \mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\ &= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - 0 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Now we normalize:

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{u}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{u}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

□

6. (10 Points) Consider the three points  $(1, 1)$ ,  $(2, 1)$ , and  $(3, 3)$  in the plane. Clearly these points are not on a single line. That means there do not exist  $\beta_0$  and  $\beta_1$  such that

$$\begin{aligned}\beta_0 + 1 \cdot \beta_1 &= 1 \\ \beta_0 + 2 \cdot \beta_1 &= 1 \\ \beta_0 + 3 \cdot \beta_1 &= 3\end{aligned}$$

(The notation is a little odd to think about lines, but at least makes a vector  $\beta$ . Notice that  $\beta_0$  is the “y-intercept” of the supposed line and  $\beta_1$  its slope).

Find  $\hat{\beta}_0$  and  $\hat{\beta}_1$  which form the least-squares solution to this system of linear equations. You will have found the line which fits this data the closest.

*Solution.* The matrix equation is of the form  $X\beta = \mathbf{y}$ , where

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

The normal equations of this system are formed by multiplying this by  $X^T$ :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

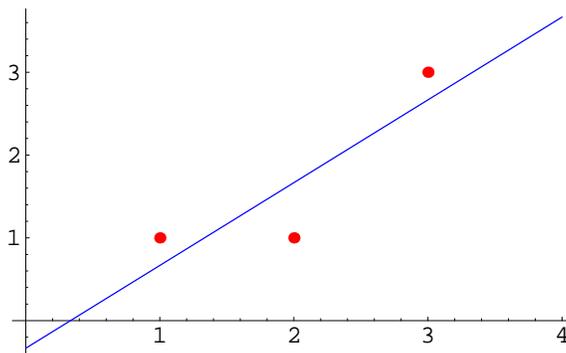
or,

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}.$$

The  $2 \times 2$  matrix is invertible, and we can solve:

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 12 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 12 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}.$$

So the line  $y = x - \frac{1}{3}$  fits the data the best.



# 6

# 6

Some went the long way around and tried to project  $\mathbf{y}$  onto the column space of  $X$ . Unfortunately, the columns of  $X$  aren't orthogonal. Therefore we have to do the projection with

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_2} \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

So

$$\hat{\mathbf{y}} = \frac{\mathbf{v}_1 \cdot \mathbf{y}}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{y}}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 = \frac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 5/3 \\ 8/3 \end{bmatrix}.$$

But this is not the answer. We still need to find  $\hat{\boldsymbol{\beta}}$  such that  $X\hat{\boldsymbol{\beta}} = \hat{\mathbf{y}}$ . We can row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 2/3 \\ 1 & 2 & 5/3 \\ 1 & 3 & 8/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So  $\hat{\beta}_0 = -1/3$  and  $\hat{\beta}_1 = 1$ , as above.

□

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