Name: ______________________________ ID#: ____________________

Solutions to
Midterm III

Math 20
Introduction to Linear Algebra
and Multivariable Calculus

December 17, 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> —*Handbook for Students*

1. (10 Points) *Let α and β be constants such that $0 < \alpha < 1$ and $\beta < 1$. Let $u(x, y)$ be the function*

$$u(x,y) = \left[\alpha x^\beta + (1-\alpha)y^\beta\right]^{1/\beta}.$$

Show that

$$\sigma \equiv \frac{u_x u_y}{u\, u_{xy}}$$

is a constant.

Solution. This is just a test of partial derivatives and some algebra. We have

$$\begin{aligned}
u_x &= \frac{1}{\beta}\left[\alpha x^\beta + (1-\alpha)y^\beta\right]^{1/\beta-1} \alpha\beta x^{\beta-1} \\
&= \left[\alpha x^\beta + (1-\alpha)y^\beta\right]^{1/\beta-1} \alpha x^{\beta-1} \\
u_y &= \frac{1}{\beta}\left[\alpha x^\beta + (1-\alpha)y^\beta\right]^{1/\beta-1} (1-\alpha)\alpha\beta y^{\beta-1} \\
&= \left[\alpha x^\beta + (1-\alpha)y^\beta\right]^{1/\beta-1} (1-\alpha)\alpha y^{\beta-1} \\
u_x y = (u_x)_y &= \left(\frac{1}{\beta} - 1\right)\left[\alpha x^\beta + (1-\alpha)y^\beta\right]^{1/\beta-2} \alpha x^{\beta-1}(1-\alpha)y^{\beta-1}
\end{aligned}$$

Putting this all together, we have

$$\begin{aligned}
\sigma &= \frac{\left[\alpha x^\beta + (1-\alpha)y^\beta\right]^{1/\beta-1} \alpha x^{\beta-1} \left[\alpha x^\beta + (1-\alpha)y^\beta\right]^{1/\beta-1} (1-\alpha)\alpha y^{\beta-1}}{\left[\alpha x^\beta + (1-\alpha)y^\beta\right]^{1/\beta} \left(\frac{1}{\beta} - 1\right)\left[\alpha x^\beta + (1-\alpha)y^\beta\right]^{1/\beta-2} \alpha x^{\beta-1}(1-\alpha)y^{\beta-1}} \\
&= \frac{1}{\frac{1}{\beta} - 1} = \frac{1}{1-\beta}
\end{aligned}$$

Incidentally, u is known as the *constant elasticity of substitution* function because σ is (a simplified form of) the elasticity of the marginal rate of substitution (u_x/u_y) with respect to the intensity of x (x/y). □

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2. (15 Points) *Let $S \subset \mathbb{R}^3$ be the surface of points (x, y, z) such that*

$$\left[(x-4)^2+y^2\right]\left[(x+4)^2+y^2\right] = z^4.$$

Find the equation of the plane which is tangent to S at the point $(5, 0, 3)$. Your final answer should be of the form

$$ax + by + cz = d, \qquad (*)$$

where a, b, c, and d are numbers, possibly zero.

Solution. There are two ways to do this. The way suggested during the exam was to consider z as an implicitly defined function of x and y. This means we can differentiate $(*)$ with respect to x and y:

$$\left[(x-4)^2+y^2\right](2)(x+4) + (2)(x-4)\left[(x+4)^2+y^2\right] = 4z\frac{\partial z}{\partial x}$$

At the point $(5, 0, 3)$,

$$\begin{aligned}[1](2)(9) + (2)(1)[81] &= 4(27)\frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial x} &= \frac{5}{3}.\end{aligned}$$

Similarly,

$$\left[(x-4)^2+y^2\right](2y) + (2y)\left[(x+4)^2+y^2\right] = 4z\frac{\partial z}{\partial y}$$

At the point we are considering, $y = 0$ and the left-hand side is zero. Hence $\dfrac{\partial z}{\partial y} = 0$.

This means the linear approximation to z at the point $(5, 0, 3)$ is

$$\begin{aligned}z - z_0 &= \left.\frac{\partial z}{\partial x}\right|_{x_0,y_0,z_0}(x-x_0) + \left.\frac{\partial z}{\partial y}\right|_{x_0,y_0,z_0}(y-y_0) \\ z - 3 &= \frac{5}{3}(x-5)\end{aligned}$$

So the equation in standard form is

$$5x - 3z = 16.$$

□

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3. (21 Points) *Find and classify all critical points of the function*

$$f(x, y) = 2y^2x + 3x^2y - 4xy.$$

Hint. There are four critical points. If you can't find them all, make sure you classify the ones you do find.

Solution. The critical or stationary points are the places where both partial derivatives vanish.

$$\frac{\partial f}{\partial x} = 2y^2 + 6xy - 4y$$
$$\frac{\partial f}{\partial y} = 4yx + 3x^2 - 4x$$

So (x, y) is a critical point if and only if

$$2y(y + 3x - 2) = 0 \tag{3.1}$$
$$x(4y + 3x - 4) = 0. \tag{3.2}$$

To solve (3.1), we have two possibilities. One of these is $y = 0$. This transforms (3.2) into $x(3x - 4)$, which has solutions $x = 0$ or $x = \frac{4}{3}$. Thus we have two critical points: $(0, 0)$ and $(\frac{4}{3}, 0)$.

Another solution to (3.1) is $y = 2-3x$. Plugging this into (3.2) gives $x(4-9x) = 0$, so either $x = 0$ or $x = \frac{4}{9}$. This gives $y = 2$ or $y = \frac{2}{3}$. Therefore we have two more critical points: $(0, 2)$ and $(\frac{4}{9}, \frac{2}{3})$.

We need to find the Hessian at each of these critical points:

$$H(x, y) = \begin{bmatrix} 6y & 6x + 4y - 4 \\ 6x + 4y - 4 & 4x \end{bmatrix}$$

They are:

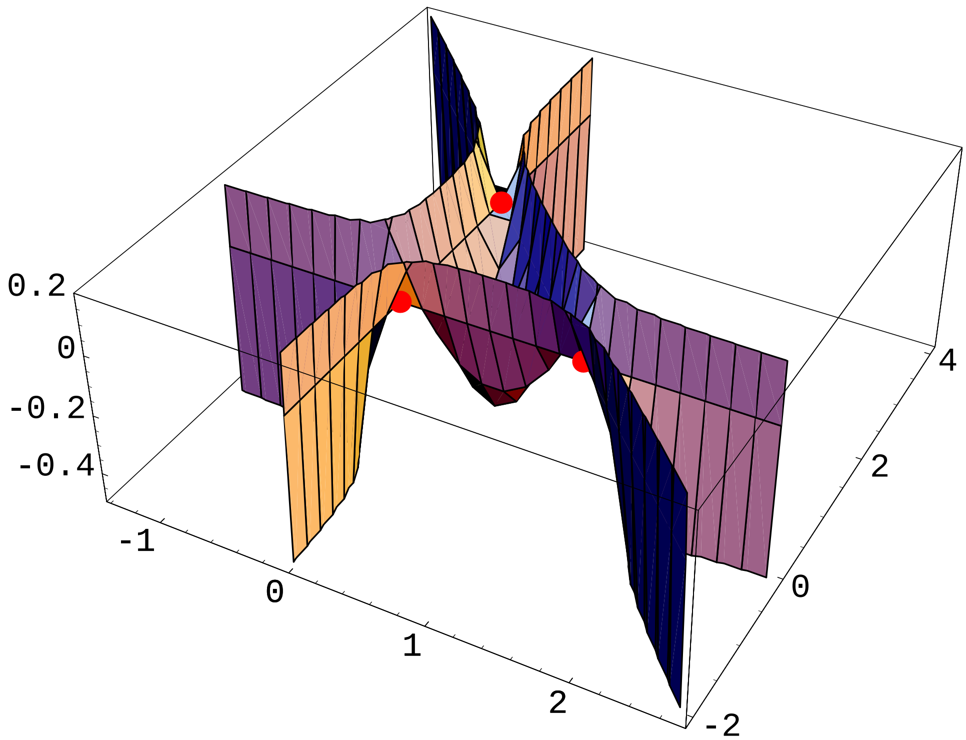
$$H(0, 0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} \qquad H(\tfrac{4}{3}, 0) = \begin{bmatrix} 0 & 4 \\ 4 & \frac{16}{3} \end{bmatrix}$$
$$H(\tfrac{4}{9}, \tfrac{2}{3}) = \begin{bmatrix} 4 & \frac{4}{3} \\ \frac{4}{3} & \frac{16}{9} \end{bmatrix} \qquad H(0, 2) = \begin{bmatrix} 12 & 4 \\ 4 & 0 \end{bmatrix}$$

The determinants of $H(0, 0)$, $H(\frac{4}{3}, 0)$, and $H(0, 2)$ are all negative becuase of the zeroes on the diagonals. Thus these three critical points are all saddles. For the remaining critical point:

$$\det H(\tfrac{4}{9}, \tfrac{2}{3}) = \tfrac{64}{9} - \tfrac{16}{9} = \tfrac{48}{9} = \tfrac{16}{3} > 0$$
$$\operatorname{tr} H(\tfrac{4}{9}, \tfrac{2}{3}) = 4 + \tfrac{16}{9} > 0$$

Therefore this point is a local minimum.

Here is a plot of f with the saddle points shown. The local minimum is in the middle of them.



□

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4. (13 Points) *A Harvard Student Agencies company sells 100 anti-Yale T-Shirts every year. The cost of keeping one in stock is \$1/year, and there is a \$2 charge on every order delivered from the distributor. It can be shown (this means you don't have to) that the cost of keeping an inventory of x units replenished y times per year is*

$$C(x, y) = \tfrac{1}{2}x + 2y.$$

What x and y will minimize costs while satisfying demand for the 100 T-Shirts (i.e., $xy = 100$)? What is this minimum cost?

Solution. The method of Lagrange multipliers says to insert a variable λ and form the equations $\nabla f = \lambda \nabla g$, $g = 100$, where $g(x, y) = xy$. This gives three equations in the three variables x, y, λ:

$$\tfrac{1}{2} = \lambda y \tag{4.1}$$

$$2 = \lambda x \tag{4.2}$$

$$xy = 100. \tag{4.3}$$

Solving (4.1) for x and (4.2) for y, and plugging both into (4.3) gives

$$\frac{1}{\lambda^2} = 100$$

$$\implies \lambda = \tfrac{1}{10}.$$

Now we know by (4.1) and (4.2) that

$$x = 20 \qquad\qquad y = 5$$

Therefore the minimum cost is

$$C(20, 5) = \tfrac{1}{2}(20) + 2(5) = 20.$$

□

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