

In-class problems on partial differentiation

Math 20

1 December 2004

Find the first partial derivatives of the functions.

1. $f(x, y) = 3x - 2y^4$

Solution. We have

$$\frac{\partial f}{\partial x} = 3 \qquad \frac{\partial f}{\partial y} = -8y^3$$

▲

2. $w = \sin \alpha \cos \beta$

Solution. We have

$$\frac{\partial w}{\partial \alpha} = \cos \alpha \cos \beta \qquad \frac{\partial w}{\partial \beta} = -\sin \alpha \sin \beta$$

▲

3. $f(u, v) = \arctan(u/v)$

Solution. For this it's important to remember the chain rule!

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{1}{1 + (u/v)^2} \frac{\partial}{\partial u} \frac{u}{v} = \frac{1}{1 + (u/v)^2} \frac{1}{v} \\ \frac{\partial f}{\partial v} &= \frac{1}{1 + (u/v)^2} \frac{\partial}{\partial v} \frac{u}{v} = \frac{1}{1 + (u/v)^2} \frac{-u}{v^2} \end{aligned}$$

Another way to write this is

$$\frac{\partial f}{\partial u} = \frac{u}{u^2 + v^2} \qquad \frac{\partial f}{\partial v} = \frac{-v}{u^2 + v^2}$$

▲

4. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Solution. We have a partial derivative for each index i , but luckily they're symmetric. So each derivative is represented by:

$$\begin{aligned}\frac{\partial u}{\partial x_i} &= \frac{1}{2\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}} \frac{\partial}{\partial x_i} (x_1^2 + x_2^2 + \dots + x_n^2) \\ &= \frac{x_i}{\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}}\end{aligned}$$

▲

Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$:

5. $x^2 + y^2 - z^2 = 2x(y + z)$

Solution. It's important to remember that implicit differentiation assumes z is dependent on x and y but x and y are independent. Therefore:

$$\begin{aligned}2x + 0 - 2z \frac{\partial z}{\partial x} &= 2(y + z) + 2x \frac{\partial z}{\partial x} \\ \implies \frac{\partial z}{\partial x} &= \frac{x - y - z}{x + z}\end{aligned}$$

$$\begin{aligned}0 + 2y - 2z \frac{\partial z}{\partial y} &= 2x \left(1 + \frac{\partial z}{\partial y}\right) \\ \implies \frac{\partial z}{\partial y} &= \frac{y - x}{x + z}\end{aligned}$$

▲

6. $xyz = \cos(x + y + z)$

Solution.

$$\begin{aligned}yz + xy \frac{\partial z}{\partial x} &= -\sin(x + y + z) \frac{\partial z}{\partial x} \\ \implies \frac{\partial z}{\partial x} &= -\frac{yz}{xy + \sin(x + y + z)}\end{aligned}$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{xz}{xy + \sin(x + y + z)}$$

▲

Find all the second partial derivatives.

7. $f(x, y) = x^4 - 3x^2y^3$

Solution. We have

$$\frac{\partial f}{\partial x} = 4x^3 - 6xy^3 \qquad \frac{\partial f}{\partial y} = -9x^2y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 6y^3 \qquad \frac{\partial^2 f}{\partial x \partial y} = -18xy^2 \qquad \frac{\partial^2 f}{\partial y^2} = -18x^2y$$

▲

8. $u = e^{-s} \sin t$.

Solution.

$$\frac{\partial u}{\partial s} = -e^{-s} \sin t \qquad \frac{\partial u}{\partial t} = e^{-s} \cos t$$

$$\frac{\partial^2 u}{\partial s^2} = e^{-s} \sin t \qquad \frac{\partial^2 u}{\partial s \partial t} = -e^{-s} \cos t \qquad \frac{\partial^2 u}{\partial t^2} = -e^{-s} \sin t$$

▲