⑦ maximize/minimize $f(x,y,z) = 2x + 6y + 10z$ such that $x^2+y^2+z^2 = 35$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = 35$

$\nabla f(x,y,z) = (2, 6, 10)$ $\nabla g(x,y,z) = (2x, 2y, 2z)$

$$\left.\begin{array}{l} \text{i) } 2 = \lambda \cdot 2x \\ \text{ii) } 6 = \lambda \cdot 2y \\ \text{iii) } 10 = \lambda \cdot 2z \\ \text{iv) } x^2+y^2+z^2 = 35 \end{array}\right\}$$

case 1: $\lambda = 0$

$\Rightarrow$ contradiction since from (i) we get $2 = 0 \cdot (2x)$

case 2: $\lambda \neq 0$

from (i), (ii), (iii) we have $\frac{1}{\lambda} = x = \frac{1}{3}y = \frac{1}{5}z \Rightarrow \begin{array}{l} 3x = y \\ 5x = z \end{array}$

plug into (iv): $x^2 + (3x)^2 + (5x)^2 = 35$

$35x^2 = 35 \Rightarrow x = \pm 1$

if $x = 1$: $y = 3x = 3$ and $z = 5x = 5$

if $x = -1$: $y = 3x = -3$ and $z = 5x = -5$

two possible cases: $(1,3,5)$, $(-1,-3,-5)$

$f(1,3,5) = 2(1) + 6(3) + 10(5) = 70$

$f(-1,-3,-5) = 2(-1) + 6(-3) + 10(-5) = -70$

thus, $\boxed{\begin{array}{l} \text{maximum value } f(1,3,5) = 70 \\ \text{minimum value } f(-1,-3,-5) = -70 \end{array}}$

⑩ maximize/minimize $f(x,y,z) = x^2y^2z^2$ such that $x^2+y^2+z^2 = 1$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = 1$

$\nabla f(x,y,z) = (2xy^2z^2, 2x^2yz^2, 2x^2y^2z)$ $\nabla g(x,y,z) = (2x, 2y, 2z)$

$$\left.\begin{array}{l} \text{i) } 2xy^2z^2 = \lambda \cdot 2x \\ \text{ii) } 2x^2yz^2 = \lambda \cdot 2y \\ \text{iii) } 2x^2y^2z = \lambda \cdot 2z \\ \text{iv) } x^2+y^2+z^2 = 1 \end{array}\right\}$$

case 1: from (i) we get $x(\lambda - y^2z^2) = 0$

thus, either $x = 0$ or $\lambda = y^2z^2$

if $x = 0$: by inspection of $f(x,y,z)$ we know this is a minimum since the function can never be less than zero

from (ii), (iii) we have $0 = \lambda y = \lambda z$

if $\lambda = 0$: from (iv) we have $y^2+z^2 = 1$, so any combination of (x,y,z) such that $x = 0$ and $y^2+z^2 = 1$ is a minimum

if $y = 0$ or $z = 0$: this case is encompassed by statement above

if $\lambda = y^2z^2$: from (ii) we have $x^2yz^2 = y^3z^2$ and from (iii) we have $x^2y^2z = y^2z^3$

simplifying: $(x^2 - y^2)yz^2 = 0$ and $(x^2 - z^2)y^2z = 0$

we've already taken care of $y = 0$ and $z = 0$ cases so ignore those

if $x^2 = y^2$: from (iii) we have $(y^2 - z^2)y^2z = 0 \Rightarrow y^2 = z^2$

from (iv) we have $y^2+y^2+y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{3}}$

since this problem is symmetric, we would get same result for x and z