(10) (con't) possible cases: $\left(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right), \left(\frac{1}{\sqrt3},\frac{1}{\sqrt3},-\frac{1}{\sqrt3}\right), \left(\frac{1}{\sqrt3},-\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right), \left(-\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right)$

$\left(\frac{1}{\sqrt3},-\frac{1}{\sqrt3},-\frac{1}{\sqrt3}\right), \left(-\frac{1}{\sqrt3},\frac{1}{\sqrt3},-\frac{1}{\sqrt3}\right), \left(-\frac{1}{\sqrt3},-\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right), \left(-\frac{1}{\sqrt3},-\frac{1}{\sqrt3},-\frac{1}{\sqrt3}\right)$ } 8 cases

$f\left(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right) = \left(\frac{1}{\sqrt3}\right)^6 = \frac{1}{27}$ ← this is the same for all cases listed above

thus, maximum value $f(x,y,z) = \frac{1}{27}$ for each of above cases

minimum value $f(x,y,z) = 0$ for any case where at least one of x,y,z equals 0, and satisfies $x^2+y^2+z^2=1$
(note, $x=y=z=0$ doesn't satisfy conditions)

(21) maximize $P = bL^\alpha K^{1-\alpha}$ subject to budget constraint $mL+nK=p$

$\nabla P(L,K) = \lambda \nabla B(L,K)$ and $B(L,K)=p$

$\nabla P(L,K) = \left(b\alpha L^{\alpha-1}K^{1-\alpha},\ b(1-\alpha)L^\alpha K^{-\alpha}\right)$ $\qquad \nabla B(L,K) = (m,n)$

i) $b\alpha L^{\alpha-1}K^{1-\alpha} = \lambda m$
ii) $b(1-\alpha)L^\alpha K^{-\alpha} = \lambda n$
iii) $mL+nK=p$

(*note $L^{\alpha-1}K^{1-\alpha} = \left(\frac{K}{L}\right)^{1-\alpha}$
*note $L^\alpha K^{-\alpha} = \left(\frac{L}{K}\right)^\alpha$)

case 1: $\lambda=0$

from (i), (ii) we have $0 = b\alpha\left(\frac{K}{L}\right)^{1-\alpha} = b(1-\alpha)\left(\frac{L}{K}\right)^\alpha$
where we know $1-\alpha>0$ and $\alpha>0$ (since $0<\alpha<1$)
thus, this is inconsistent since the first equality gives $K=0$, but this makes second part undefined

case 2: $\lambda\neq0$

from (i), (ii) we have $\lambda = \frac{b\alpha}{m}L^{\alpha-1}K^{1-\alpha} = \frac{b(1-\alpha)}{n}L^\alpha K^{-\alpha}$

simplifying, $L = \frac{\alpha n}{(1-\alpha)m}K$

plug into (iii), $\cancel{m}\left(\frac{\alpha n}{(1-\alpha)\cancel{m}}\right)K + nK = p$

solve for K, $\left(\frac{\alpha}{1-\alpha}+1\right)K = \frac{p}{n} \Rightarrow K = \frac{(1-\alpha)p}{n}$

plug back for L, $mL + \cancel{n}\left(\frac{(1-\alpha)p}{\cancel{n}}\right) = p$

Solve for L, $L = \frac{1}{m}(p-(1-\alpha)p) = \frac{\alpha p}{m}$

thus, maximum production P occurs with $L = \frac{\alpha p}{m}$ and $K = \frac{(1-\alpha)p}{n}$