(19)

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -7 & 1 \\ 0 & \lambda-1 & 0 \\ 0 & -15 & \lambda+2 \end{vmatrix} = (\lambda+1)\left[(\lambda-1)(\lambda+2)\right] + 7\left[0\right] + 1\left[0\right] = (\lambda+1)(\lambda-1)(\lambda+2) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = -2$$

$\lambda_1 = 1$
$$\begin{bmatrix} 2 & -7 & 1 \\ 0 & 0 & 0 \\ 0 & -15 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} 2x_1 - 7x_2 + x_3 = 0 \\ -5x_2 + x_3 = 0 \end{array} \Bigg\} \begin{array}{c} x_3 = 5x_2 \\ x_1 = x_2 = t \end{array} \qquad P_1 = \begin{bmatrix} t \\ t \\ 5t \end{bmatrix}$$

$\lambda_2 = -1$
$$\begin{bmatrix} 0 & -7 & 1 \\ 0 & -2 & 0 \\ 0 & -15 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} -7x_2 + x_3 = 0 \\ -x_2 = 0 \end{array} \Bigg\} \begin{array}{c} x_2 = x_3 = 0 \\ x_1 = t \end{array} \qquad P_2 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_3 = -2$
$$\begin{bmatrix} -1 & -7 & 1 \\ 0 & -3 & 0 \\ 0 & -15 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} -x_1 - 7x_2 + x_3 = 0 \\ -3x_2 = 0 \end{array} \Bigg\} \begin{array}{c} x_2 = 0 \\ x_1 = x_3 = t \end{array} \qquad P_3 = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 5 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 0 \\ -1 & -4 & 5 \\ 0 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & -5 & 1 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A^n = PD^n P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} (1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-2)^n \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2^n \\ 1 & 0 & 0 \\ 5 & 0 & -2^n \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & -5 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}$$

(21)

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-3 \end{vmatrix} = (\lambda-3)\left[(\lambda-2)(\lambda-3)-1\right] - 1\left[\lambda-3\right] + 0 = (\lambda-3)\left[\lambda^2 - 5\lambda + 5\right] - (\lambda-3) = (\lambda-3)(\lambda^2-5\lambda+4)$$

$$= (\lambda-3)(\lambda-1)(\lambda-4) = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda_3 = 4$$

$\lambda_1 = 1$:
$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} -2x_1 + x_2 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 2x_3 = 0 \end{array} \Bigg\} \begin{array}{c} x_2 = 2x_1 \\ x_1 = x_3 = t \end{array} \qquad P_1 = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix}$$

$\lambda_2 = 3$
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} x_2 = 0 \\ x_1 - x_2 + x_3 = 0 \end{array} \Bigg\} \; x_1 = -x_3 = t \qquad P_2 = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix}$$

$\lambda_3 = 4$
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} x_1 + x_2 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ x_2 + x_3 = 0 \end{array} \Bigg\} \; x_1 = -x_2 = x_3 = t \qquad P_3 = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{-6}\begin{bmatrix} -1 & -3 & -2 \\ -2 & 0 & 2 \\ -1 & 3 & -2 \end{bmatrix}^T = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \Rightarrow D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$