Math 20 Spring 2005
Final Exam Practice Problems (Set 1)

1. Consider the following system of linear equations.

$$\begin{array}{rcrcl} x & + & 3y & = & k \\ 4x & + & hy & = & 8 \end{array}$$

Find values of h and k such that the system...

(a) ...has no solutions.
(b) ...has a unique solution.
(c) ...has infinitely many solutions.

2. Suppose

$$A = QR = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 & 0 \\ 0 & 3 & -6 & -8 \\ 0 & 0 & 4 & -9 \\ 0 & 0 & 0 & 7 \end{bmatrix},$$

where Q is a 5×4 matrix with columns $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, and $\mathbf{q}_4$. Write an expression that gives the third column of A.

3. Suppose an $n \times n$ matrix A satisfies the equation

$$A^2 - 2A + I = 0.$$

(a) Express A^3 as a linear combination of A and I.
(b) Find A^{-1}.

4. Find the standard matrix for the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that projects a vector onto the line $2x - y = 0$.

5. Suppose A is an $m \times n$ matrix whose columns are linearly independent.

(a) If for some $\mathbf{b}$, a solution of $A\mathbf{x} = \mathbf{b}$ exists, is the solution unique? Why or why not?
(b) What can you say about the relation between the numbers m and n for this matrix A?
(c) What can you say about m and n if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbf{R}^m$?

6. Assume that the matrices A and B shown below are row equivalent.

$$A = \begin{bmatrix} 2 & 4 & 1 & 6 & 0 \\ 3 & 6 & 2 & 7 & -3 \\ 2 & 4 & 0 & 10 & 4 \\ 3 & 6 & 2 & 7 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 & 5 & 2 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the rank of A? How did you find this?
(b) Find a basis for the subspace spanned by the columns of A.

7. Diagonalize the following matrix, if possible. If not possible, explain why. (You may use the fact that the eigenvalues of the matrix are $\lambda = 1$ and $\lambda = -2$.)

$$\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

8. Suppose that $T(x, y)$ gives the temperature at x degree latitude and y degrees longitude at 7 pm on May 17, 2005, where $-90 \leq x \leq 90$ and $-180 \leq y \leq 180$.
   (a) For a fixed point (a, b), which is likely to be greater $T_x(a, b)$ or $T_y(a, b)$ and why?
   (b) For what values of x would you expect $\frac{\partial T}{\partial x}$ to be positive? negative?
   (c) Suppose that for some point (a, b), $T_x(a, b) = T_y(a, b) = 0$. What can you say about the temperature at the point (a, b)?

9. Find an equation of the tangent plane to the surface $z = x^2 + xy + 3y^2$ at the point $(1, 1, 5)$.

10. What is the maximum value of the function $f(x, y) = xy$ assuming that $x^2 + 4y^2 = 1$?

11. Find the extreme values of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the region $\{(x, y) \mid x^2 + y^2 \leq 16\}$.

12. Match each of the following functions with its graph and with its contour map.

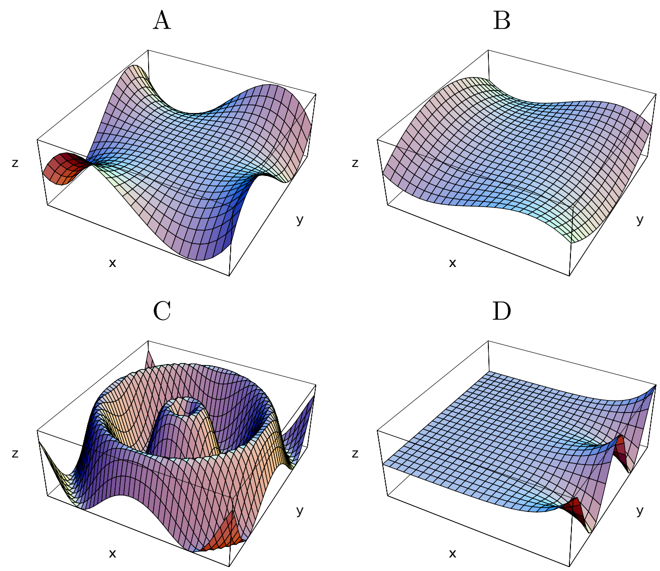
(a) $f(x, y) = e^x \cos y$

(b) $f(x, y) = xy^2 - x^3$

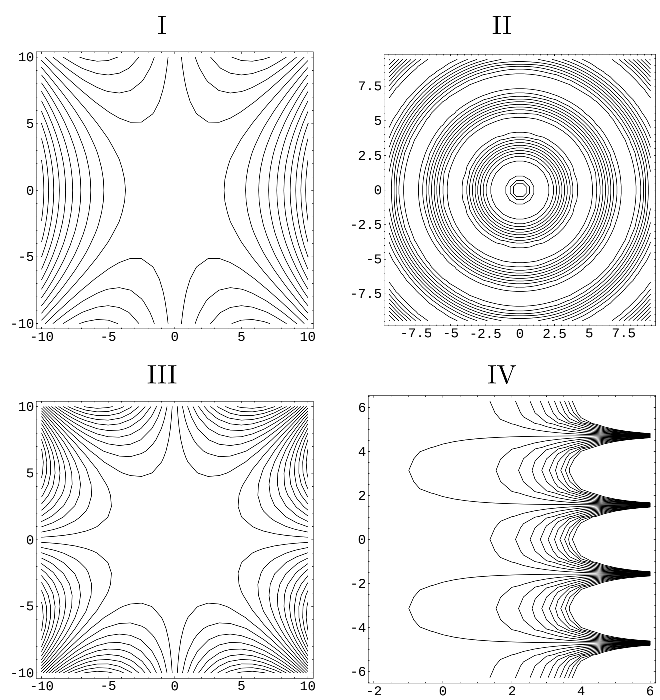
(c) $f(x, y) = xy^3 - x^3 y$

(d) $f(x, y) = \sin \sqrt{x^2 + y^2}$

Graphs



Contour Maps



Answers

1. (a) Note that the slope of the line $x + 3y = k$ is $-\frac{1}{3}$. For the system to have no solutions, the lines must be parallel and distinct. Since the slope of the line $4x + hy = 8$ is $-\frac{4}{h}$, it follows that h must equal 12 for the lines to be parallel. If $h = 12$ and $k = 2$, then the second line is identical to the first one, so for the system to have no solutions we must have $h = 12$ and $k \neq 2$.

   (b) In order for the system to have a unique solution, the lines must not be parallel. Thus $h \neq 12$ and k can be anything.

   (c) In order to have infinitely many solutions, the lines must be identical. As shown in part (a), this occurs when $h = 12$ and $k = 2$.

2. Note that

$$A = QR = Q\begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4\end{bmatrix} = \begin{bmatrix}Q\mathbf{r}_1 & Q\mathbf{r}_2 & Q\mathbf{r}_3 & Q\mathbf{r}_4\end{bmatrix},$$

where $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4$ are the columns of R. It follows that the third column of A is the vector

$$Q\mathbf{r}_3 = \begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4\end{bmatrix}\begin{bmatrix}5\\-6\\4\\0\end{bmatrix} = 5\mathbf{q}_1 - 6\mathbf{q}_2 + 4\mathbf{q}_3.$$

3. (a) Note that

$$A^2 = 2A - I$$

   and so

$$A^3 = A(2A - I) = 2A^2 - A = 2(2A - I) - A = 4A - 2I - A = 3A - 2I.$$

   (b) Note that

$$I = 2A - A^2 = A(2I - A).$$

   Since A times $2I - A$ equals the identity matrix, it follows that $A^{-1} = 2I - A$.

4. Recall that the standard matrix of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ is given by $\begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2)\end{bmatrix}$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis for $\mathbf{R}^2$. Using similar triangles, one can show that $T(\mathbf{e}_1) = \begin{bmatrix}\frac{1}{5}\\\frac{2}{5}\end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix}\frac{2}{5}\\\frac{4}{5}\end{bmatrix}$. Thus the standard matrix for T is $\begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\\frac{2}{5} & \frac{4}{5}\end{bmatrix}$. (Note that this problem is a little more difficult than one I would ask on the final exam, due to the creative use of similar triangles needed to solve it.)

5. (a) Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions to the equation $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$ and so $A\mathbf{x}_1 = A\mathbf{x}_2$. This implies that

$$\mathbf{0} = A\mathbf{x}_1 - A\mathbf{x}_2 = A(\mathbf{x}_1 - \mathbf{x}_2).$$

   Since the columns of A are linearly independent, the only solution to the homogeneous equation $\mathbf{0} = A\mathbf{x}$ is the trivial solution, and so $\mathbf{x}_1 - \mathbf{x}_2$ must equal the zero vector. Thus $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$ and so $\mathbf{x}_1 = \mathbf{x}_2$. This argument shows that there can be only one distinct solution to the equation $A\mathbf{x} = \mathbf{b}$.

(b) Since the columns of A are linearly independent, A must have a pivot position in every column. Thus A has n pivot positions, since n is the number of columns of A. It follows that A must have at least this many rows, and so m must be greater than or equal to n.

(c) If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$, then A must have a pivot position in every row. Thus A must have m pivot positions. Since we already know A has n pivot positions, it follows that $m = n$.

6. (a) The rank of A is the dimension of its columns space which is given by the number of pivot columns of A. Since A and B are row equivalent, they have the same pivot columns. Inspection of B shows that the matrices have 3 pivot columns, so the rank of A is 3.

   (b) A basis for the column space of A is given by the pivot columns of A, which correspond to the pivot columns of B:

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 4 \\ 9 \end{bmatrix} \right\}.$$

7. To find the eigenspace corresponding to the eigenvalue $\lambda = 1$, we must solve the equation $(A - 1I)\mathbf{x} = \mathbf{0}$. This gives the augmented matrix

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ -4 & -7 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that $x_1 = x_3$ and $x_2 = -x_3$ and so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Thus the eigenspace corresponding to $\lambda = 1$ is spanned by a single vector, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

To find the eigenspace corresponding to the eigenvalue $\lambda = -2$, we must solve the equation $(A - (-2)I)\mathbf{x} = \mathbf{0}$. This gives the augmented matrix

$$\begin{bmatrix} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that $x_1 = -x_2$ and $x_3 = 0$ and so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Thus the eigenspace corresponding to $\lambda = 1$ is spanned by a single vector, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Since the matrix is 3×3 but the sum of the dimensions of its eigenspaces is only 2, the matrix is not diagonalizable.

8. (a) Since $T_x(a, b)$ gives the rate of change in temperature as latitude (x) increases from south to north and $T_y(a, b)$ gives the rate of change in temperature as longitude (y) increases from west to east, we would expect $T_x(a, b)$ to be bigger since temperature typically changes more as one travels south to north than west to east.

   At least, that's what I said in the review session. That's not completely correct. Let's look at part (b) and then return to part (a).

   (b) Note that $\frac{\partial T}{\partial x}$ gives the rate of change in temperature as one travels from south to north. Since temperature generally increases as one travels from the South Pole to the equator, one would expect $\frac{\partial T}{\partial x}$ to be positive when $-90 < x < 0$. Since temperature generally decreases as one travels from the equator to the North Pole, one would expect $\frac{\partial T}{\partial x}$ to be negative when $0 < x < 90$.

   Now let's return to part (a). Suppose the point (a, b) were in the southern hemisphere. Then we would expect $T_x(a, b)$ to be positive and $T_y(a, b)$ to be close to zero. Thus $T_x(a, b)$ would indeed be greater than $T_y(a, b)$. However, if the point (a, b) were in the northern hemisphere, $T_x(a, b)$ would be *negative* and $T_y(a, b)$ would be close to zero. Thus $T_x(a, b)$ would be *less than* $T_y(a, b)$. No matter where (a, b) is, we would expect $|T_x(a, b)|$ to be greater than $|T_y(a, b)|$. This is a more accurate answer to part (a).

   (c) For such a point, the rate of change in temperature as you travel from south to north through the point is zero, as is the rate of change in temperature as you travel from west to east through the point. It follows that the temperature is roughly constant near the point (a, b).

   The point (a, b) is a critical point of the temperature function. It may be that (a, b) has the highest temperature nearby (a local max), the lowest temperature nearby (a local min), or neither (a saddle point).

9. An equation of the tangent line to the function $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

   In this case, $f_x(x, y) = 2x + y$ and $f_y(x, y) = x + 6y$ and so $f_x(1, 1) = 3$ and $f_y(1, 1) = 7$. Thus an equation of the tangent line is

$$z = 3(x - 1) + 7(y - 1) + 5.$$

10. We can use the method of Lagrange multipliers to solve this problem. We want to maximize the function $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + 4y^2 = 1$. This maximum must occur at a point (x, y) that satisfies the two equations

$$\frac{f_x}{g_x} = \frac{f_y}{g_y} \qquad \text{and} \qquad g(x, y) = k,$$

   where $k = 1$ in this case.

   Computing partial derivatives, we find that $f_x = y$, $f_y = x$, $g_x = 2x$, and $g_y = 8y$. Thus, we want to solve the following two equations:

$$\frac{y}{2x} = \frac{x}{8y} \qquad \text{and} \qquad x^2 + 4y^2 = 1.$$

The first equation gives us $8y^2 = 2x^2$ and so $x^2 = 4y^2$. Substituting this into the second equation, we get $4y^2 + 4y^2 = 1$, which implies that $y = \pm\frac{1}{2\sqrt{2}}$. Since $x^2 = 4y^2$, we find that $x = \pm\frac{1}{\sqrt{2}}$. This gives us four candidates for the maxima of f subject to the given constraint. We now evaluate f at each of these candidates to see which candidate(s) give us the greatest value.

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = \frac{1}{4}$$

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = -\frac{1}{4}$$

$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = -\frac{1}{4}$$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = \frac{1}{4}$$

Thus f achieves its maximum value of $\frac{1}{4}$ at the points $(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}})$.

11. I will go over this problem at our next review session.

12. (a) D, IV
    (b) B, I
    (c) A, III
    (d) C, II