1. Find the extreme values of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the region $\{(x, y) \mid x^2 + y^2 \le 16\}$.

2. **Allocation of Funds:** A new editor has been allotted $60,000 to spend on the development and promotion of a new book. It is estimated that if x thousand dollars is spent on development and y thousand on promotion, approximately $f(x, y) = 20x^{3/2}y$ copies of the book will be sold. How much money should the editor allocate to development and how much to promotion in order to maximize sales?

3. **Marginal Analysis:** Suppose the editor in the earlier problem is allotted $61,000 instead of $60,000 to spend on the development and promotion of the new book. Estimate how the additional $1,000 will affect the maximum sales level.
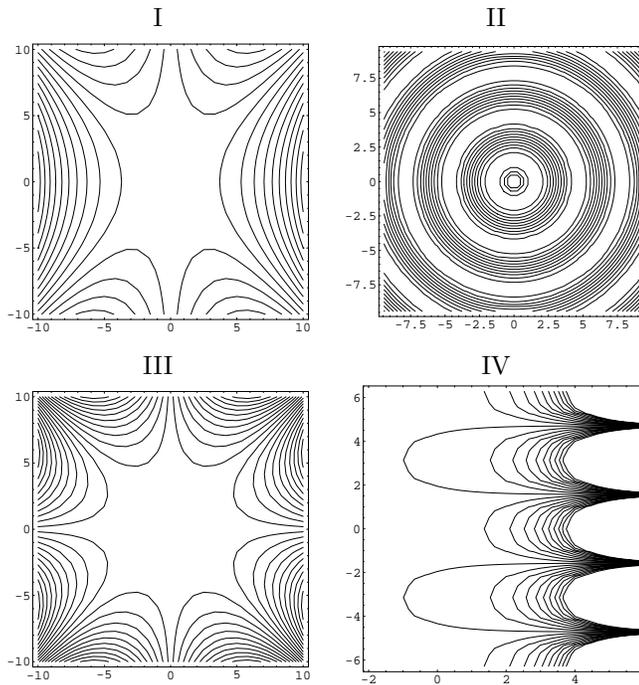
4. Match each of the following functions with its graph and with its contour map.

(a) $f(x, y) = e^x \cos y$

(b) $f(x, y) = xy^2 - x^3$

(c) $f(x, y) = xy^3 - x^3 y$
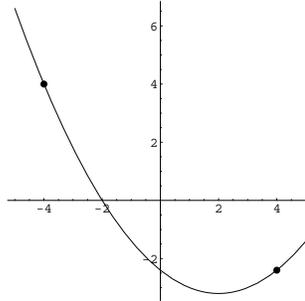
(d) $f(x, y) = \sin \sqrt{x^2 + y^2}$

Graphs

A

B

C

D

Contour Maps

I

II

III

IV
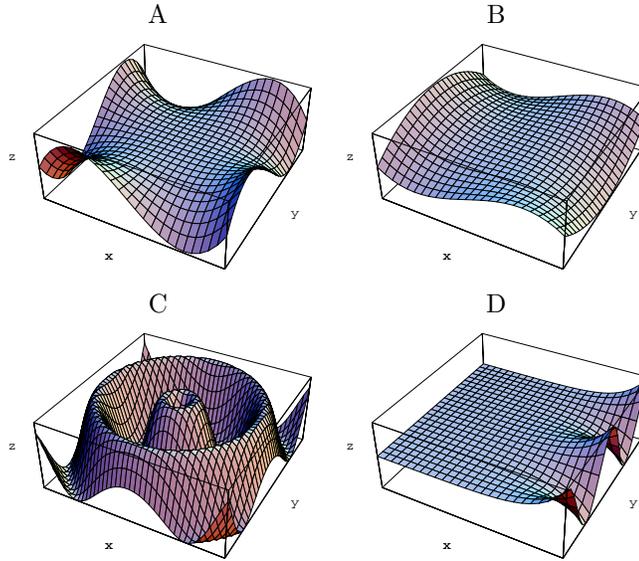
Answers

1. Here's a similar problem from first-semester calculus: Find the extreme values of the function $f(x) = \frac{1}{5}(x^2 - 4x - 12)$ on the interval $[-4, 4]$.

The extreme values might occur at the endpoints of the interval or in the interior of the interval. If an extreme value occurs in the interior of the interval, then it must occur at a critical point. For the above function, the absolute maximum occurs at the endpoint $x = -4$ and the absolute minimum occurs at the critical point $x = 2$. To find these extreme, you would have to first find the critical points and then evaluate the function at all critical points and endpoints. We will use a similar strategy for the two-variable problem at hand.

The extreme values of the function $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the region $\{(x, y) \mid x^2 + y^2 \leq 16\}$ will occur either on the interior of the region or on the boundary of the region. That is, they will either occur on the interior $\{(x, y) \mid x^2 + y^2 < 16\}$ or on the boundary $\{(x, y) \mid x^2 + y^2 = 16\}$.

Step One: Determine the critical points of the function on the interior of the region. Note that $f_x = 4x - 4$ and $f_y = 6y$. Setting both of these equal to zero, we find that the only critical point is $x = 1$, $y = 0$. The point $(1, 0)$ is in the interior $\{(x, y) \mid x^2 + y^2 < 16\}$, so it is a candidate for the extreme values of this function on the region $\{(x, y) \mid x^2 + y^2 \leq 16\}$.

Step Two: If the extreme values do not occur at the critical point of the function, then they must occur at points on the boundary of the region, $\{(x, y) \mid x^2 + y^2 = 16\}$. Since there are infinitely many such points, we don't want to test them all! We'll use the technique of Lagrange multipliers to determine which of the boundary points are extreme points on the boundary.

This means that we want to identify the extreme points of the function $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ subject to the constraint $x^2 + y^2 = 16$. This gives us our function g: $g(x, y) = x^2 + y^2 - 16$. The extreme points will occur at (x, y) pairs satisfying the following two equations.

$$\frac{f_x}{g_x} = \frac{f_y}{g_y} \qquad \text{and} \qquad g(x, y) = 0$$

Simplifying the first equation, we get

$$\frac{f_x}{g_x} = \frac{f_y}{g_y}$$
$$\frac{4x - 4}{2x} = \frac{6y}{2y}$$
$$(4x - 4)(2y) = (6y)(2x)$$
$$(4x - 4)(2y) - (6y)(2x) = 0$$
$$(4y)((x - 1)(2) - (3)(x)) = 0$$
$$(4y)(-x - 2) = 0$$

which implies that either $y = 0$ or $x = -2$.

- *Case One:* If $y = 0$, then we can substitute that in the g equation to get

$$0 = x^2 + y^2 - 16 = x^2 + 0^2 - 16 = x^2 - 16$$

  which implies that $x = \pm 4$. This gives us two candidates for extreme values: $(-4, 0)$ and $(4, 0)$.

- *Case Two:* If $x = -2$, then we can substitute that in the g equation to get

$$0 = x^2 + y^2 - 16 = (-2)^2 + y^2 - 16 = y^2 - 12$$

  which implies that $y = \pm\sqrt{12}$. This gives us two more candidates for extreme values: $(-2, \sqrt{12})$ and $(-2, -\sqrt{12})$.

Step Three: We now have five candidates for extreme values. We'll substitute each of them into f to see which give the greatest outputs and which give the least outputs.

$$f(1, 0) = -7$$
$$f(-4, 0) = 43$$
$$f(4, 0) = 11$$
$$f(-2, \sqrt{12}) = 47$$
$$f(-2, -\sqrt{12}) = 47$$

Thus, the function obtains its maximum value of 47 at the points $(-2, \sqrt{12})$ and $(-2, -\sqrt{12})$ and it obtains is minimum value of -7 at the point $(1, 0)$.

2. Note that we want to maximize the function $f(x, y) = 20x^{3/2}y$ subject to the constraint $x + y = 60$. This gives us our g function: $g(x, y) = x + y - 60$. The extreme points will occur at (x, y) pairs satisfying the following two equations.

$$\frac{f_x}{g_x} = \frac{f_y}{g_y} \qquad \text{and} \qquad g(x, y) = 0$$

Simplifying the first equation, we get

$$\frac{f_x}{g_x} = \frac{f_y}{g_y}$$
$$\frac{30x^{1/2}y}{1} = \frac{20x^{3/2}}{1}$$
$$30x^{1/2}y = 20x^{3/2}$$
$$30x^{1/2}y - 20x^{3/2} = 0$$
$$10x^{1/2}(3y - 2x) = 0$$

which implies that either $x = 0$ or $y = \frac{2}{3}x$.

- *Case One:* If $x = 0$, then we can substitute that in the g equation to get

$$0 = x + y - 60 = 0 + y - 60$$

  which implies that $y = 60$. This gives us one candidate for the maximum value: $(0, 60)$.

- *Case Two:* If $y = \frac{2}{3}x$, then we can substitute that in the g equation to get

$$0 = x + y - 60 = x + \frac{2}{3}x - 60 = \frac{5}{3}x - 60$$

  which implies that $x = 36$. Since $y = \frac{2}{3}x$, we get that $y = \frac{2}{3}36 = 24$. This gives us another candidate for the maximum value: $(36, 24)$.

We'll substitute each of these into f to see which give the greatest output.

$$f(0, 60) = 0$$
$$f(36, 24) = 103,680$$

Thus, we can maximize sales by spending \$36,000 on development and \$24,000 on promotion, yielding sales of 103,680 books.

3. Recall the following two items from the lecture notes on Lagrange multipliers.

   - **Lagrange Multipliers:** Given a point (x, y) that satisfies the two equations

     $$\frac{f_x}{g_x} = \frac{f_y}{g_y} \qquad \text{and} \qquad g(x, y) = 0,$$

     let

     $$\lambda = \frac{f_x}{g_x} = \frac{f_y}{g_y}.$$

     We call λ a Lagrange multiplier.

   - **Theorem:** Suppose M is the maximum (or minimum) value of $f(x, y)$ subject to the constraint $g(x, y) = k$. The Lagrange multiplier λ is the rate of change of M with respect to k. That is,

     $$\lambda = \frac{dM}{dk}.$$

     Hence λ approximates the change in M resulting from a 1-unit increase in k.

   In this problem M is the maximum sales corresponding to a budget of k thousand dollars. Note that the instantaneous rate of change in M with respect to k is approximately equal to the average rate of change in M with respect to k:

   $$\frac{dM}{dk} \approx \frac{\Delta M}{\Delta k},$$

   assuming Δk is relatively small. This gives us that

   $$\lambda = \frac{dM}{dk} \approx \frac{\Delta M}{\Delta k}$$

   or

   $$\Delta M \approx \lambda \Delta k.$$

   This tells us that if we change the budget by Δk thousand dollars, then the change in maximum sales is approximately $\lambda \Delta k$. In this example, $\Delta k = 1$ and

   $$\lambda = \frac{f_x}{g_y}\bigg|_{x=36, y=24} = \frac{30x^{1/2}y}{1}\bigg|_{x=36, y=24} = 30(36)^{1/2}(24) = 4320.$$

   Thus, by increasing the budget by one thousand dollars, we can expect to sell 4,320 more books.

4. (a) D, IV
   (b) B, I
   (c) A, III
   (d) C, II