1. Suppose that a group of three people, Larry, Moe, and Curly, work together on a farm. All of the food produced by these three people is consumed among the three. In addition, the amount of food produced by each person equals the total amount consumed by that person. (For example, Larry eats the same amount of food that he produces.)

   Of the food produced by Larry, $\frac{1}{3}$ is consumed by Larry himself and $\frac{2}{3}$ is consumed by Moe. Of the food produced by Moe, $\frac{1}{2}$ is consumed by Larry and $\frac{1}{2}$ is consumed by Curly. Of the food produced by Curly, $\frac{1}{4}$ is consumed by Moe and $\frac{3}{4}$ is consumed by Curly himself.

   Assuming that Curly consumes 32 pounds of food a week, how much food do Larry and Moe consume a week?

2. The network in the figure below shows a proposed plan for the traffic flow around a new park. The plan calls for a computerized traffic light at the north exit on Third Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the park. All streets are one-way.

   (a) How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the network is the same as the average number of vehicles flowing out?

   (b) Assuming that the traffic light has been set to balance the total flow in and out of the network, what are upper and lower bounds on the average number of vehicles per hour that will flow along each of the streets that border the park?
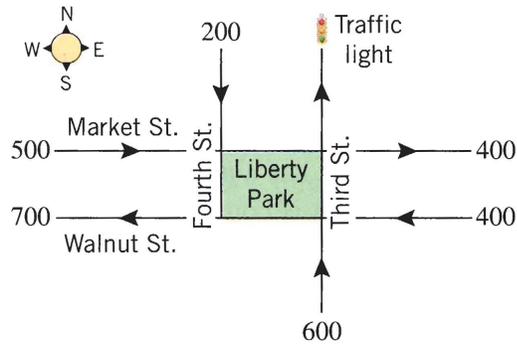
3. Two friends (we'll call them Player R and Player C) play a modified version of the two-person game rock-paper-scissors. Each player chooses one of rock, paper, or scissors. Instead of using the usual rock-paper-scissors rules, they assign points each round according to the following payoff matrix.

$R \setminus C$	Rock	Paper	Scissors
Rock	2	-2	0
Paper	-6	0	-5
Scissors	5	2	3

   (a) If Player R chooses paper 50% of the time and scissors 50% of the time, and if Player C chooses rock 75% of the time and scissors 25% of the time, which player will score more points on average? Justify your answer.

   (b) If each player chooses an optimal strategy, which player will score more points on average? Justify your answer.

4. The weather in Columbus is either good, indifferent, or bad on any given day. If the weather is good today, there is a 60% chance the weather will be good tomorrow, a 30% chance the weather will be indifferent, and a 10% chance the weather will be bad. If the weather is indifferent today, it will be good tomorrow with probability .40 and indifferent with probability .30. Finally, if the weather is bad today, it will be good tomorrow with probability .40 and indifferent with probability .50.

   (a) Suppose there is a 50% chance of good weather today and a 50% chance of indifferent weather. What are the chances of bad weather tomorrow?

   (b) What are the chances of good, indifferent, and bad weather many days from now?

5. Denote by c_k and m_k, respectively, the cat population and mouse population in a small town in month k. Suppose that
$$
\begin{aligned}
c_{k+1} &= 0.7c_k + 0.2m_k \\
m_{k+1} &= -0.6c_k + 1.4m_k
\end{aligned}
$$
   for $k = 0, 1, 2, \ldots$.

   (a) Suppose that $3c_0 = 2m_0$. Use an eigenvalue decomposition to determine the long term growth rate in the cat/mouse population and the long term proportion of cats to mouse.

   (b) Suppose that $3c_0 < 2m_0$. Use an eigenvalue decomposition to determine the long term growth rate in the cat/mouse population and the long term proportion of cats to mouse.

6. A cell phone manufacturer is introducing two new lines of phones, the Standard and the Executive. Each phone needs a battery, a chip, and a plastic case.

   (a) The Standard model uses Everlast I batteries, available in unlimited quantities from an outside supplier. The Executive model uses Everlast II batteries, 6,000 of which can be produced in-house each year.

   (b) The Executive model uses Intel 60 chips, available in unlimited quantities from an outside supplier. The Standard model uses Intel 48 chips, 4,000 of which can be produced in-house each year.

   (c) The Standard model's plastic case requires 3 ounces of plastic, while the Executive model's plastic case requires 2 ounces of plastic. The cell phone manufacturer has access to 18,000 ounces of plastic each year.

   (d) Market research indicates that the Standard model can be sold for $3 net profit, while the Executive model can be sold for $5 net profit.

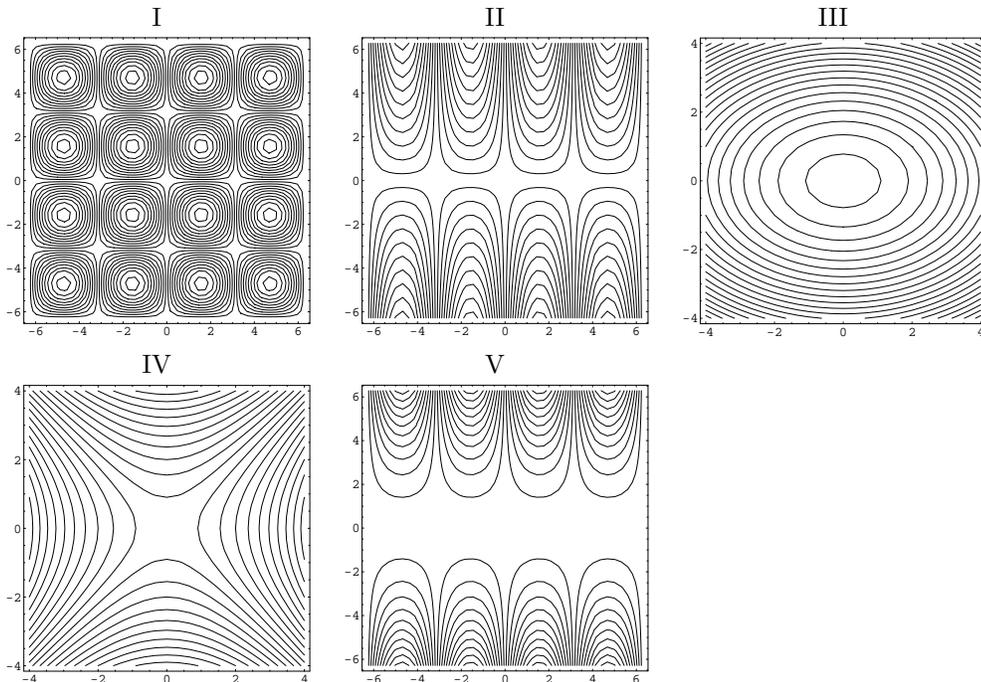   What product mix will maximize expected profit for the coming year?

7. Match each of the following functions with its graph and with its contour map.

Function	Graph	Contour Map
(a) $f(x,y) = x^2 + y^2$	——————	——————
(b) $f(x,y) = x^2 - y^2$	——————	——————
(c) $f(x,y) = \sin x \sin y$	——————	——————
(d) $f(x,y) = y^2 \sin x$	——————	——————
(e) $f(x,y) = y \sin x$	——————	——————

Graphs

A   B   C

D   E

Contour Maps

I   II   III

IV   V
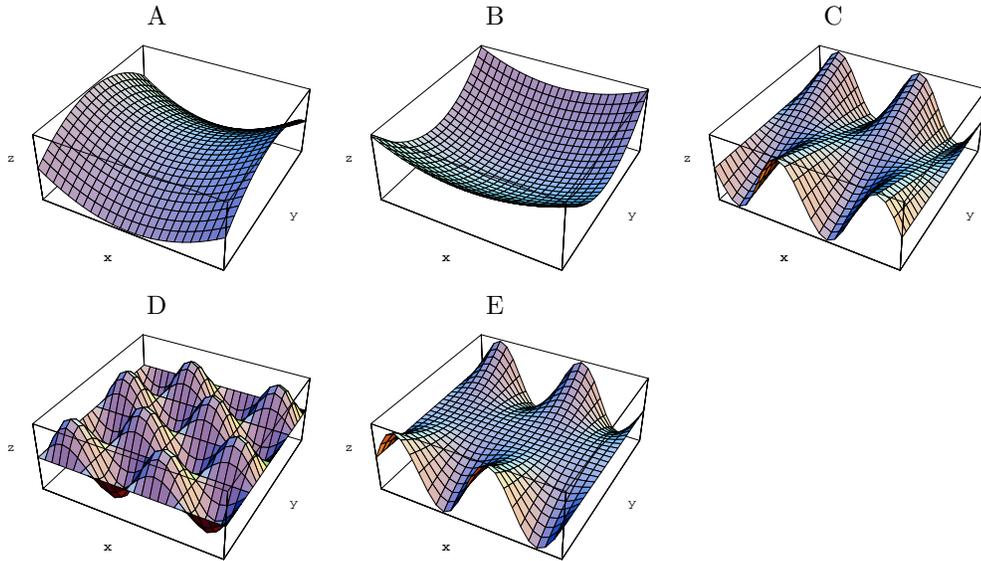
Answers

1. Let L, M, and C be the number of pounds of food produced by Larry, Moe, and Curly, respectively. Since Larry consumes $\frac{1}{3}$ of his own food and $\frac{1}{2}$ of Moe's food, it follows that Larry consumes $\frac{1}{3}L + \frac{1}{2}M$ pounds of food. Since Larry produces L pounds and he produces as much as he consumes, we have the following equation.

$$L = \frac{1}{3}L + \frac{1}{2}M$$

Similar analyses produce equations for Moe and Curly:

$$M = \frac{2}{3}L + \frac{1}{4}C$$

$$C = \frac{1}{2}M + \frac{3}{4}C$$

This gives us the following system of three linear equations.

$$
\begin{array}{rcrcrcr}
L & = & \frac{1}{3}L & + & \frac{1}{2}M & + & 0C \\
M & = & \frac{2}{3}L & + & 0M & + & \frac{1}{4}C \\
C & = & 0L & + & \frac{1}{2}M & + & \frac{3}{4}C
\end{array}
$$

Collecting all variables on one side, we get the following.

$$
\begin{array}{rcrcrcr}
-\frac{2}{3}L & + & \frac{1}{2}M & + & 0C & = & 0 \\
\frac{2}{3}L & - & M & + & \frac{1}{4}C & = & 0 \\
0L & + & \frac{1}{2}M & - & \frac{1}{4}C & = & 0
\end{array}
$$

Knowing that Curly consumes 32 pounds of food simplifies the system since $C = 32$.

$$
\begin{array}{rcrcr}
-\frac{2}{3}L & + & \frac{1}{2}M & = & 0 \\
\frac{2}{3}L & - & M & = & -8 \\
0L & + & \frac{1}{2}M & = & 8
\end{array}
$$

Thus $M = 16$ and $L = \frac{3}{2}(M - 8) = 12$. Thus Moe produces 16 pounds of food and Larry produces 12 pounds of food.

2. (a) Since the flow into the entire system should equal the flow out of the entire system, we have

$$500 + 600 + 400 + 200 = 700 + 400 + x$$

where x equals the number of vehicles exiting the system north on Third Street. Thus, $x = 600$, so the traffic light should let 600 vehicles through per hour.

(b) Let x_1, x_2, x_3, and x_4 be the average number of vehicles per hour on the streets adjacent to the park.
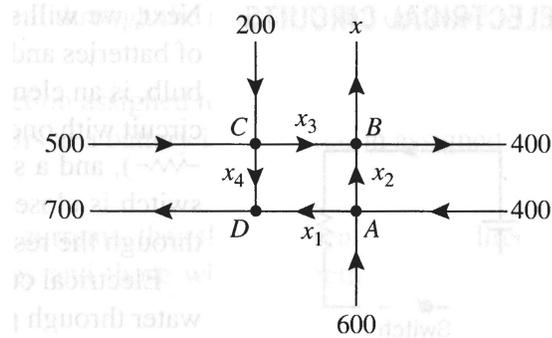
Since the flow in to each intersection must equal the flow out of each intersection, we get the following four equations, one for each intersection.

$$
\begin{array}{rcl}
400 + 600 &=& x_1 + x_2 \\
x_2 + x_3 &=& 400 + x \\
500 + 200 &=& x_3 + x_4 \\
x_1 + x_4 &=& 700
\end{array}
$$

With $x = 600$, as computed in part (a), we get the following linear system.

$$
\begin{array}{rcl}
x_1 + x_2 &=& 1000 \\
x_2 + x_3 &=& 1000 \\
x_3 + x_4 &=& 700 \\
x_1 + x_4 &=& 700
\end{array}
$$

This system has solution

$$
\begin{aligned}
x_1 &= 700 - x_4 \\
x_2 &= 300 + x_4 \\
x_3 &= 700 - x_4 \\
x_4 &\text{ free.}
\end{aligned}
$$

Note that all four variables must be greater than or equal to zero. Thus

$$
700 - x_4 \geq 0
$$
$$
300 + x_4 \geq 0
$$

These inequalities imply that $x_4 \leq 700$. This gives us the following lower and upper bounds for the traffic flow on each street.

$$
0 \leq x_1 \leq 700
$$
$$
300 \leq x_2 \leq 1000
$$
$$
0 \leq x_3 \leq 700
$$
$$
0 \leq x_4 \leq 700
$$

3. (a) We are given the payoff matrix

$$
A = \begin{bmatrix} 2 & -2 & 0 \\ -6 & 0 & -5 \\ 5 & 2 & 3 \end{bmatrix}.
$$

Given that Player R uses the strategy

$$
\mathbf{p} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}
$$

and Player C uses the strategy

$$\mathbf{q} = \begin{bmatrix} 0.75 \\ 0 \\ 0.25 \end{bmatrix},$$

then the expected value of the game is

$$\mathbf{p}^T A \mathbf{q} = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -6 & 0 & -5 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0 \\ 0.25 \end{bmatrix} = -0.625.$$

Since the expected value of the game is negative, we can expect that Player R will lose more points on average.

(b) Since the 2 in the third row and second column of the payoff matrix is the smallest entry in its row and the largest entry in its column, it is a saddle point of the payoff matrix and so it determines the optimal strategies for both players. Player R should choose scissors all of the time, and Player C should choose paper all of the time. When they do, Player R will score 2 points each round, and so Player R will win more points on average.

4. (a) We have the following transition matrix.

| | Today's Weather | | |
Tomorrow's Weather	Good	Indifferent	Bad
Good	.6	.4	.4
Indifferent	.3	.3	.5
Bad	.1	.3	.1

Given an initial state vector of

$$\mathbf{x}_0 = \begin{bmatrix} .5 & .5 & 0 \end{bmatrix}$$

the subsequent state vector is

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \begin{bmatrix} .5 & .5 & 0 \end{bmatrix} = \begin{bmatrix} .5 & .3 & .2 \end{bmatrix}.$$

Thus, there is a 20% chance of bad weather tomorrow.

(b) To find the steady state vector, we must solve the equation $A\mathbf{x} = \mathbf{x}$ or, equivalently, $(A - I)\mathbf{x} = \mathbf{0}$. This gives us the following augmented matrix.

$$\begin{bmatrix} -.4 & .4 & .4 & 0 \\ .3 & -.7 & .5 & 0 \\ .1 & .3 & -.9 & 0 \end{bmatrix}$$

This matrix has the following reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives us the solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Setting $x_3 = \frac{1}{6}$, we get the probability vector $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$. Thus, we can expect good weather 50% of

the time, indifferent weather roughly 33% of the time, and bad weather roughly 16% of the time.

5. Let $\mathbf{x}_k = \begin{bmatrix} c_k \\ m_k \end{bmatrix}$. Then we know that $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where $A = \begin{bmatrix} 0.7 & 0.2 \\ -0.6 & 1.4 \end{bmatrix}$.

According to the Eigenvector Decomposition Theorem (see the lecture notes on discrete dynamical systems), the long-term cat/mouse population vector is given by

$$\mathbf{x}_k = A^k \mathbf{x}_0 = a_1 \lambda_1^k \mathbf{v}_1 + a_2 \lambda_2^k \mathbf{v}_2,$$

where λ_1 and λ_2 are the eigenvalues of A, $\mathbf{v}_1$ and $\mathbf{v}_2$ are corresponding eigenvectors, and

$$\mathbf{x}_0 = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2.$$

To find the eigenvalues of A, we first find

$$A - \lambda I = \begin{bmatrix} 0.7 - \lambda & 0.2 \\ -0.6 & 1.4 - \lambda \end{bmatrix}.$$

We then set the determinant of this matrix equal to 0 and solve for λ:

$$0 = \det(A - \lambda I) = (0.7 - \lambda)(1.4 - \lambda) - (0.2)(-0.6) = \lambda^2 - 2.1\lambda + 1.1 = (\lambda - 1.1)(\lambda - 1)$$

Thus $\lambda_1 = 1.1$ and $\lambda_2 = 1$.

To find the eigenvectors corresponding to $\lambda_1 = 1.1$, we solve the equation $(A - 1.1I)\mathbf{x} = \mathbf{0}$. This gives us the following augmented matrix.

$$\begin{bmatrix} -0.4 & 0.2 & 0 \\ -0.6 & 0.3 & 0 \end{bmatrix}$$

This row reduces to

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which gives us that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

Thus, we can set $\mathbf{v}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

To find the eigenvectors corresponding to $\lambda_2 = 1$, we solve the equation $(A - 1I)\mathbf{x} = \mathbf{0}$. This gives us the following augmented matrix.

$$\begin{bmatrix} -0.3 & 0.2 & 0 \\ -0.6 & 0.4 & 0 \end{bmatrix}$$

This row reduces to

$$\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which gives us that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (2/3)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

Thus, we can set $\mathbf{v}_2 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$.

Thus
$$\mathbf{x}_k = A^k \mathbf{x}_0 = a_1 \lambda_1^k \mathbf{v}_1 + a_2 \lambda_2^k \mathbf{v}_2 = a_1 (1.1)^k \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + a_2 (1)^k \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

We now need to determine a_1 and a_2. Note that
$$\mathbf{x}_0 = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2$$

or
$$\begin{bmatrix} c_0 \\ m_0 \end{bmatrix} = a_1 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

or
$$\begin{bmatrix} 1/2 & 2/3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} c_0 \\ m_0 \end{bmatrix}.$$

This gives us the augmented matrix
$$\begin{bmatrix} 1/2 & 2/3 & c_0 \\ 1 & 1 & m_0 \end{bmatrix}$$

which reduces to
$$\begin{bmatrix} 1 & 0 & -2(3c_0 - 2m_0) \\ 0 & 1 & 3(2c_0 - m_0) \end{bmatrix}.$$

Thus $a_1 = -2(3c_0 - 2m_0)$ and $a_2 = 3(2c_0 - m_0)$ and so
$$\mathbf{x}_k = -2(3c_0 - 2m_0)(1.1)^k \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + 3(2c_0 - m_0)(1)^k \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

(a) If $3c_0 = 2m_0$, then
$$\mathbf{x}_k = -2(0)(1.1)^k \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + 3(2c_0 - m_0)(1)^k \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = 3(2c_0 - m_0) \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

Note that there is now no k in the formula for $\mathbf{x}_k$. Thus the cat and mouse population remains constant. The eigenvector $\mathbf{v}_2$ gives us the proportion of cats to mice: There are 3 mice for every 2 cats.

(b) If $3c_0 < 2m_0$, then the expression $-2(3c_0 - 2m_0)$ is nonzero, so $\lambda_1 = 1.1$ will be the dominant eigenvalue. Since $-2(3c_0 - 2m_0)$ will also be positive, we have that the cat and mouse population will be growing at a rate of approximately 10% per year. The eigenvector $\mathbf{v}_1$ gives us the proportion of cats to mice: There are 2 mice for every 1 cat.

Note that this problem is definitely harder than any single problem on your final exam will be! However, if you can understand this problem, then you should be set for the discrete dynamical system material.

6. Let x be the number of Standard models produced and let y be the number of Executive models produced. Then we want to maximize the profit function $p(x, y) = 3x + 5y$ subject to a number of constraints.

Since we only have access to 6,000 batteries for the Executive models, we have that
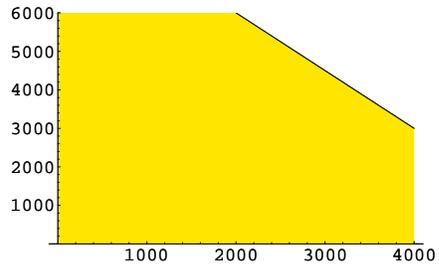$$y \le 6,000.$$

Since we only have access to 4,000 chips for the Standard models, we have that
$$x \le 4,000.$$

Since we only have access to 18,000 ounces of plastic, we have that
$$3x + 2y \le 18,000.$$

These are our three constraints, and we can use them to graph the feasible region.

We can then solve for the extreme points of the feasible region: $(0, 0)$, $(0, 6000)$, $(2000, 6000)$, $(4000, 3000)$, and $(4000, 0)$. Substituting each of these into the profit function, we find

$$p(0, 0) = 0$$
$$p(0, 6000) = 30,000$$
$$p(2000, 6000) = 36,000$$
$$p(4000, 3000) = 37,000$$
$$p(4000, 0) = 12,000$$

and so profit is maximized by producing 4,000 Standard models and 3,000 Executive models.

7. (a) B and III
   (b) A and IV
   (c) D and I
   (d) E and V
   (e) C and II