Math 20 Spring 2005
Final Exam Review Guide

1 Linear Algebra Topics

§1.1 Systems of Linear Equations

- To translate a system of linear equations into augmented matrix notation and vice versa.
- To determine whether a given equation is linear or not.
- To know which row operations can be applied to the augmented matrix of a system of linear equations without changing its solution set.
- To visualize the solution sets of systems of two or three variables.

§1.2 Row Reduction and Echelon Forms

- To determine if a given matrix is in row echelon or reduced row echelon form.
- To determine the pivot positions and pivot columns of a given matrix.
- To use elementary row operations to reduce a given matrix to row echelon or reduced row echelon form.
- To solve a system of linear equations using the row reduction algorithm, describing the solution set parametrically if appropriate.
- To determine the basic and free variables of a system of linear equations.
- To determine if a system of linear equations has no solutions, exactly one solution, or infinitely many solutions.
- To determine if a linear system is consistent by analyzing the row echelon form of its augmented matrix.

§1.3 Vector Equations

- To simplify expressions involving sums, differences, and scalar multiples of vectors.
- To visualize vectors and sums of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- To determine whether a given vector is a linear combination of a given set of vectors either algebraically or geometrically.
- To translate a system of linear equations from augmented matrix notation to vector equation notation and vice versa.
- To visualize and describe the span of a set of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

§1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

- To compute the matrix-vector product $A\mathbf{x}$.
- To translate a system of linear equations among augmented matrix notation, vector equation notation, and matrix equation notation.
- To simplify expressions involving matrix-vector products.

§1.5 Solution Sets of Linear Equations

- To express solutions to systems of linear equations in parametric vector form.
- To visualize and describe the solution set of a system of linear equations in two or three dimensions.
- To describe the relationship between the solution set of a non-homogeneous system and the solution set of the corresponding homogeneous system.

§1.7 Linear Independence

- To determine whether a given set of vectors is linearly independent, using the definition of linear independence or some other appropriate technique.
- To find a linear dependence relation among a set of linearly dependent vectors.

§1.8 Introduction to Linear Transformations

- To determine the domain, codomain, and range of a given linear transformation.
- To determine algebraically and geometrically the effect of a given linear transformation on given vectors.
- To determine if a given transformation is linear.

§1.9 The Matrix of a Linear Transformation

- To find the standard matrix of a given linear transformation given its action on the standard basis for $\mathbb{R}^n$.
- To visualize and describe the action of certain linear transformations, including projections, shears, contractions, dilations, rotations, and reflections.
- To determine if a given linear transformation is onto or one-to-one.
- To analyze the existence and uniqueness of solutions to the matrix equation $A\mathbf{x} = \mathbf{b}$ in terms of the linear transformation whose standard matrix is A.

§2.1 Matrix Operations

- To simplify expressions involving sums, differences, scalar products, matrix products, and transpositions of matrices.
- To explain in terms of linear transformations why matrix multiplication is defined as it is.
- To identify ways in which matrix multiplication is not like numerical multiplication.

§2.2 The Inverse of a Matrix

- Given two matrices, to verify that they are inverses.
- To simplify expressions involving sums, differences, scalar products, matrix products, transpositions, and *inverses* of matrices.
- To identify a given matrix as an elementary matrix and find its inverse.
- To illustrate how one can use elementary matrices to find the inverse of a matrix.
- To use row reduction to find the inverse of a matrix.

§2.3 Characterizations of Invertible Matrices

- To know and be able to apply the Invertible Matrix Theorem. (See the full Invertible Matrix Theorem later in this review guide.)

§2.8 Subspaces of $\mathbb{R}^n$

- To determine algebraically or geometrically if a given set of vectors is a subspace of $\mathbb{R}^n$.
- To determine whether a given vector is in the column space or null space of a given matrix.
- To determine if a given set of vectors is a basis for $\mathbb{R}^n$.
- To find a basis for the column space or null space of a given matrix.

§2.9 Dimension and Rank

- To find the coordinates of a given vector relative to a given basis.
- To find the dimension of a given subspace.
- To find the rank of a matrix (that is, the dimension of its columns space) or the dimension of its null space.
- To determine if a given set of vectors is a basis for a given subspace.

§5.1 Eigenvectors and Eigenvalues

- To determine geometrically the eigenvalues and eigenvectors of a linear transformation that is relatively easy to visualize, such as a reflection, rotation, projection, etc.
- To find a basis for the eigenspace of a given eigenvalue for a given matrix.
- To efficiently find the eigenvalues (and their multiplicities) of a triangular matrix.

§5.2 The Characteristic Equation

- To find the determinant of a 2×2 or 3×3 matrix.
- To simplify expressions involving the determinants of matrices.
- To find the characteristic polynomial of a given 2×2 or 3×3 or triangular matrix.
- To find the eigenvalues (and their multiplicities) of a given 2×2 or triangular matrix.

§5.3 Diagonalization

- To efficiently compute powers of a diagonal matrix.
- To efficiently compute powers of a diagonalized matrix.
- To diagonalize a matrix A by finding an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
- To determine if a given matrix is diagonalizable.

2 Application Topics

§1.6 Applications of Linear Systems

- *The Leontief Exchange Model:* Given an economy with many sectors, the total output of each sector for one time period, and how this output is divided among the other sectors, find equilibrium prices that can be assigned to the total outputs of the various sectors so that the income of each sector exactly balances its expenses.
- *Network Flow:* Given information about the flow into and out of each junction in a network, determine and analyze the general flow patten of the network.

The Assignment Problem

- To solve an assignment problem in which resources are to be optimally assigned to tasks in a one-to-one manner by modeling the problem as a matrix.
- To use the Hungarian method to solve an assignment problem.

Two-Person Zero Sum Games

- To model a two-person zero-sum game described in words with a payoff matrix.
- To determine the expected payoff of a two-person zero-sum game given its payoff matrix and particular strategies.
- To determine optimal strategies for a strictly determined two-person zero-sum game and the corresponding game value.

§2.6 The Leontief Input-Output Model

- To model a simple economy with the Leontief input-output model, finding the production levels for each of several sectors needed to satisfy both intermediate and final demand from those sectors.
- To analyze the entries of the matrix $(I - C)^{-1}$, where C is the consumption matrix in a Leontief input-output model.

§4.9 Applications to Markov Chains

- To find the transition matrix for a Markov chain given a verbal description of the Markov chain.
- To find and interpret future state vectors in a Markov chain given the transition matrix of the Markov chain and an initial state vector.
- To find and interpret the steady state vector of a Markov chain given the transition matrix of the Markov chain.

§5.6 Discrete Dynamical Systems

- To find the transition matrix for a discrete dynamical system given a verbal description of the system.
- To express the kth state vector of a discrete dynamical system as a linear combination of the eigenvectors of the system's transition matrix.
- To determine the long-term behavior of a discrete dynamical system given the eigenvalues and eigenvectors of its transition matrix.

§6.6 Applications to Linear Models

- To find the equation of the least-squares line that best fits a small set of data with one independent variable and one dependent variable.
- To describe the model that produces a least-squares fit of a set of data with one independent variable and one dependent variable by a function of a certain form.
- To describe the model that produces a least-squares fit of a set of data with two independent variables and one dependent variable by a function of a certain form.

Linear Programming and the Simplex Method

- To determine the variables, objective function, and constraints in a linear programming problem given a written description of the problem.
- To solve a linear programming problem in two variables using geometric techniques.
- To solve a linear programming problem using the simplex method.

3 Calculus Topics

LGR §9.1 Functions of Several Variables

- To evaluate a function of several variables at a given point.
- To sketch a three-dimensional graph of a linear function of two variables.
- To sketch several level curves of a function of two variables.
- To match a given function of two variables with a graph of its level curves and with a three-dimensional graph of the function.

LGR §9.2 Partial Derivatives

- To calculate the first and second partial derivatives of a function of several variables.
- To interpret the partial derivatives of a function of several variables in terms of rates of change.
- To interpret the partial derivatives of a function of two variables in terms of secant and tangent line slopes.

LGR §9.3 Maxima and Minima

- To determine the equation of the tangent plane to a given function of two variables at a given point.
- To find and classify (as local maxima, local minima, or saddle points) the critical points of a function of two variables.

LGR §9.4 Lagrange Multipliers

- To find the maximum or minimum value of a function of two variables given a constraint $g(x, y) = k$ on those variables using Lagrange multipliers.
- To estimate the rate of change in the maximum (or minimum) value of a function of two variables given a constraint $g(x, y) = k$ as k varies.

4 The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix.

(c) A has n pivot positions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA = I$.

(k) There is an $n \times n$ matrix D such that $AD = I$.

(l) A^T is an invertible matrix.

(m) The columns of A form a basis of $\mathbb{R}^n$.

(n) The column space of A is $\mathbb{R}^n$.

(o) The dimension of the column space of A is n.

(p) The rank of A is n.

(q) The null space of A equals $\{\mathbf{0}\}$.

(r) The dimension of the null space of A is 0.

(s) The number 0 is not an eigenvalue of A.

(t) The determinant of A is not zero.

5 Suggested Exercises

- §1.1 #11–17 odd
- §1.2 #3, 5, 11, 13, 15, 19, 23–31 odd, 33
- §1.3 #7, 11, 13, 15, 25, 27(a, b)
- §1.4 #7, 9, 13–21 odd, 25, 31, 33
- §1.5 #5, 9, 11, 15, 27–35 odd
- §1.6 #1, 3(a, b), 11, 13
- §1.7 #5, 7, 15, 17, 19, 23–29 odd, 33–39 odd
- §1.8 #3, 7–19 odd, 27, 31, 33, 35
- §1.9 #1–13 odd, 17, 19, 25–31 odd, 35
- Chapter 1 Supplementary Exercises #1, 7, 11, 13, 17, 19, 25(a, b)
- §2.1 #9, 11, 13, 17–23 odd
- §2.2 #11–23 odd, 29, 31, 35, 37
- §2.3 #3, 5, 7, 13–23 odd, 27, 29, 31
- §2.6 #1–9 odd
- §2.8 #5, 7, 9, 15, 17, 19, 23–35 odd
- §2.9 #1–7 odd, 13, 15, 19–25 odd
- Chapter 2 Supplementary Exercises #1–9 odd, 15, 17
- §4.9 #1–13 odd, 17
- §5.1 #3–19 odd, 25–31 odd, 35
- §5.2 #5–11 odd, 15, 17, 25, 27
- §5.3 #1, 5–19 odd, 23, 25, 27
- §5.6 #1, 3, 5, 17(a, b)
- Chapter 5 Supplementary Exercises #1, 3
- §6.6 #1, 3, 7(a), 9
- LGR §9.1 #3, 5, 7, 13, 15, 21–26, 31, 33, 35
- LGR §9.2 #11–19 odd, 21, 23, 33, 35, 47, 49
- LGR §9.3 #9–13 odd, 21, 23, 25, 33, 35, 37
- LGR §9.4 #5, 7, 19, 21, 23