Math 20 Spring 2005
Lagrange Multipliers

1 Introductory Examples

1. **Allocation of Funds:** A new editor has been allotted \$60,000 to spend on the development and promotion of a new book. It is estimated that if x thousand dollars is spent on development and y thousand on promotion, approximately $f(x, y) = 20x^{3/2}y$ copies of the book will be sold. How much money should the editor allocate to development and how much to promotion in order to maximize sales?
2. **Maximization of Utility:** A consumer has \$600 to spend on two commodities, the first of which costs \$20 per unit and the second \$30 per unit. Suppose that the utility derived by the consumer from x units of the first commodity and y units of the second commodity is given by the *Cobb-Douglas utility function* $f(x, y) = 10x^{0.6}y^{0.4}$. How many units of each commodity should the consumer buy to maximize utility?
3. **Context-Free Problem:** What is the maximum value of the function $f(x, y) = xy$ assuming that $x^2 + 4y^2 = 1$?
4. See the Mathematica notebook `lagrange.nb` for graphs for each of these problems.
5. Note that each of these problems asks you to maximize a function $f(x, y)$ subject to a constraint $g(x, y) = k$. Geometrically, this means you must find the highest level curve of f that intersects the constraint curve $g(x, y) = k$. As the Mathematica graphs suggest, the critical intersection will occur at a point where the constraint curve is tangent to a level curve; that is, at a point on the constraint curve $g(x, y) = k$ where the slope of the constraint curve is equal to the slope of the level curve $f(x, y) = C$. How can we find the slope of a constraint curve or level curve?

2 The Chain Rule in Two Variables

1. Suppose that $z = f(x, y)$. Then we know that

   $$f(u, v) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)(u - a) + \frac{\partial f}{\partial y}(a, b)(v - b)$$

   as long as the point (u, v) is near the point (a, b). Why? Because the equation of the tangent plane to $z = f(x, y)$ at the point (a, b) is

   $$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

   So

   $$f(a, b) + \frac{\partial f}{\partial x}(a, b)(u - a) + \frac{\partial f}{\partial y}(a, b)(v - b)$$

   gives the z-value of this tangent plane at the point (u, v). And since the tangent plane is a good approximation of the surface $z = f(x, y)$ near the point of tangency, we get the above equation.
2. Now suppose that both x and y are functions of some other variable t. Then $z = f(x(t), y(t))$. This means that z depends entirely on t, and so z is a function of the single variable t:

   $$z(t) = f(x(t), y(t))$$

   The question at hand is, What is $z'(t)$?

3. Let Δt be some small number and let

$$\Delta z = z(t+\Delta t) - z(t) = f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t)).$$

Let $a = x(t)$ and $b = y(t)$. Then

$$f(x(t+\Delta t), y(t+\Delta t)) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x(t+\Delta t) - a) + \frac{\partial f}{\partial y}(a,b)(y(t+\Delta t) - b).$$

Then

$$\begin{aligned} \Delta z &= f(x(t+\Delta t), y(t+\Delta t)) - f(a,b) \\ &\approx \frac{\partial f}{\partial x}(a,b)(x(t+\Delta t) - a) + \frac{\partial f}{\partial y}(a,b)(y(t+\Delta t) - b). \end{aligned}$$

and so

$$\frac{\Delta z}{\Delta t} \approx \frac{\partial f}{\partial x}(a,b)\frac{x(t+\Delta t) - a}{\Delta t} + \frac{\partial f}{\partial y}(a,b)\frac{y(t+\Delta t) - b}{\Delta t}.$$

It follows that

$$\begin{aligned} z'(t) &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\ &\approx \lim_{\Delta t \to 0} \left(\frac{\partial f}{\partial x}(a,b)\frac{x(t+\Delta t) - x(t)}{\Delta t} + \frac{\partial f}{\partial y}(a,b)\frac{y(t+\Delta t) - y(t)}{\Delta t} \right) \\ &= \frac{\partial f}{\partial x}(a,b)x'(t) + \frac{\partial f}{\partial y}(a,b)y'(t) \\ &= \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t). \end{aligned}$$

This gives us the Chain Rule for functions of two variables.

4. **The Chain Rule:** Suppose $z(t) = f(x(t), y(t))$. Then

$$z'(t) = \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t).$$

3 Lagrange Multipliers

1. Now suppose that $z(t) = f(t, y(t))$, that is, suppose that $x(t) = t$. Then

$$\begin{aligned} z'(t) &= \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t) \\ &= \frac{\partial f}{\partial x}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t))y'(t). \end{aligned}$$

Now let's change all the t's to x's. We get that $z(x) = f(x, y(x))$, that is, both z and y are functions of x. We also get that

$$z'(x) = \frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x))y'(x).$$

Now that we only have one independent variable, x, let's simplify our notation a little. We get that if $z = f(x,y)$, then

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)\frac{dy}{dx}$$

or

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

2. Now suppose you are looking at a level curve $f(x,y) = k$ for some constant k. Let's differentiate both sides with respect to x.

$$\frac{d}{dx} f(x,y) = \frac{d}{dx} k$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

What does this mean? This gives a formula for the slope of the tangent line to a level curve in terms of the partial derivatives of the function determining the level curves. Now we can find the slope of a level curve!

3. Let's return to the optimization problems from the start of the lesson. We want to find the points on the constraint curve at which the slope to the constraint curve $g(x,y) = k$ is equal to the slope of the level curve $f(x,y) = C$. Using the above formula, we have that the slope of the constraint curve is given by

$$-\frac{g_x}{g_y}$$

and the slope of the level curve is given by

$$-\frac{f_x}{f_y}.$$

Setting these equal, we get

$$-\frac{g_x}{g_y} = -\frac{f_x}{f_y}$$

or

$$\frac{f_x}{g_x} = \frac{f_y}{g_y}.$$

Since the optimum point must occur on the constraint curve, we have the following method for solving a constrained optimization problem.

4. **Constrained Optimization:** All relative extrema of the function $z = f(x,y)$ subject to a constraint $g(x,y) = 0$ will be found among those points (x,y) satisfying the equations

$$\frac{f_x}{g_x} = \frac{f_y}{g_y} \qquad \text{and} \qquad g(x,y) = 0.$$

5. **Very Important Note:** When solving problems using this technique, you should cross multiply first! That is

$$\frac{f_x}{g_x} = \frac{f_y}{g_y} \qquad \Longrightarrow \qquad f_x g_y = f_y g_x$$

Then move all terms to one side:

$$f_x g_y - f_y g_x = 0$$

Now that you have an expression equal to zero, you should attempt to factor the left-hand side.

6. **Lagrange Multipliers:** Given a point (x,y) that satisfies these two equations, let

$$\lambda = \frac{f_x}{g_x} = \frac{f_y}{g_y}.$$

We call λ a Lagrange multiplier.

7. **Theorem:** Suppose M is the maximum (or minimum) value of $f(x,y)$ subject to the constraint $g(x,y) = k$. The Lagrange multiplier λ is the rate of change of M with respect to k. That is,

$$\lambda = \frac{dM}{dk}.$$

Hence λ approximates the change in M resulting from a 1-unit increase in k.

8. **Marginal Analysis:** Suppose the editor in the earlier problem is allotted \$61,000 instead of \$60,000 to spend on the development and promotion of the new book. Estimate how the additional \$1,000 will affect the maximum sales level.