Math 20
Linear Algebra with Mathematica

Complete the exercises below as part of this week's problem set. You should turn in a printed copy of the Mathematica "notebook" (that's the document you create in the Mathematica interface) you used to complete the exercises, along with your answers to Questions 9, 10, and 13 in the Smallville problem and your answers to Questions 7 and 8 in the owl population problem. Your printed Mathematica notebook should make it clear how you completed each exercise.

See the Resources page of our course web site for more information about Mathematica.

Markov Chain Example: Smallville is the only town in its mostly rural county in Kansas. It is found that every year about 20% of those living in the town of Smallville move to the surrounding countryside and about 30% of those living in the countryside move into the town of Smallville. What percentage of the county's population can be expected to sides in the town of Smallville after many years?

1. Define the variable `P` to be the transition matrix for this Markov chain by inputing the following command.

   ```
   P = {{8/10, 3/10}, {2/10, 7/10}}
   ```

2. Have Mathematica output the transition matrix as a matrix by inputing the following command.

   ```
   MatrixForm[P]
   ```

3. Find the eigenvalues of P by inputing the following command.

   ```
   eval=Eigenvalues[P]
   ```

4. Let λ_1 be the first of these eigenvalues by inputing the following command.

   ```
   lambda1=eval[[1]]
   ```

   Let λ_2 be the second of these eigenvalues by inputing a similar command.

5. Find corresponding eigenvectors of P by inputing the following command.

   ```
   evec=Eigenvectors[P]
   ```

6. Let $\mathbf{v}_1$ be the first of these eigenvectors and $\mathbf{v}_2$ be the second by inputing the appropriate commands.

7. Suppose we knew that this year, 40% of Smallville's residents live in the town and 60% live in the country. Define the variable `x0` to be the initial state vector $\mathbf{x}_0 = \begin{bmatrix} 4/10 \\ 6/10 \end{bmatrix}$.

8. Find the weights c_1 and c_2 used to express $\mathbf{x}_0$ as a linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ by inputing the following command.

   ```
   sols = Solve[x0 == c1*v1 + c2*v2]
   ```

9. The Eigenvector Decomposition Theorem tells us that for any positive integer k,

   $$\mathbf{x}_k = P^k\mathbf{x}_0 = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2.$$

   Use the values of c_1, c_2, λ_1, λ_2, $\mathbf{v}_1$, and $\mathbf{v}_2$ you have found so far to find an explicit formula for $\mathbf{x}_k$, the population vector at time k. Do this on paper at this time, and simply the formula you find.

   The first component tells you the percentage of Smallville residents living in town at time k and the second tells you the percentage living in the country at time k.

10. What can you tell about the long term behavior of the Markov chain from this formula?

11. Generate a list of the values of each component for $k = 0$ to $k = 20$ by inputing the following command, where ??? is replaced by the formulas you found in the preceding questions.

    ```
    town = Table[???,{k,0,20}]
    country = Table[???,{k,0,20}]
    ```

12. Plot these values using the following two commands. Note that the different values for Hue produce different colored points.

    ```
    plot1=ListPlot[town,PlotRange->{0,1},PlotStyle->Hue[.1]]
    plot2=ListPlot[country,PlotRange->{0,1},PlotStyle->Hue[.6]]
    ```

13. Graph these values on a single set of axes by using the following command.

    ```
    Show[plot1,plot2]
    ```

14. What can you tell about the long term behavior of the Markov chain from this graph?

Owl Population Example: The life cycle of spotted owls divides naturally into three stages: juvenile (up to 1 year old), subadult (1 to 2 years), and adult (over 2 years). Let j_k denote the number of juveniles at year k, s_k the number of subadults at year k, and a_k the number of adults at year k. Suppose the following relationships hold among j_k, s_k, and a_k.

$$\begin{aligned} j_{k+1} &= 0.33a_k \\ s_{k+1} &= 0.18j_k \\ a_{k+1} &= 0.71s_k + 0.94a_k \end{aligned}$$

What can we say about the long-term health of the spotted owl population?

1. Define the variable A to be the transition matrix for this Markov chain.

2. Determine the eigenvalues λ_1, λ_2, and λ_3 of the transition matrix.

3. Use the Abs command to determine the magnitude of each of these eigenvalues. Note that one of them is larger in magnitude than the others. This eigenvalue is called the *dominant eigenvalue*.

4. Determine corresponding eigenvectors of A.

5. Define the variable v1 to be the eigenvector that corresponds to the dominant eigenvalue.

6. Note that the sum of the components of v1 is not 1. Scale v1 by the appropriate amount so that the sum of its components is 1 by using the following command, where ??? is the appropriate amount.

    ```
    v1 = v1/???
    ```

7. Recall from our discussion in the previous class that when k is large,

    $$\mathbf{x}_k \approx a_1\lambda_1^k\mathbf{v}_1,$$

    where a_1 is some real number, λ_1 is the dominant eigenvalue, and $\mathbf{v}_1$ is an eigenvector corresponding to λ_1. Given the values of λ_1 and $\mathbf{v}_1$ that you have found, what can you say about the long term prospects of the owl population? Be as specific as you can.

8. Now suppose that the juvenile-to-subadult survival rate was 30% instead of 18%. Repeat the analysis you have just done and determine the long term prospects of the owl population in this case. Be as specific as you can.