Math 20 Spring 2005
Midterm 1 Review Guide

1 Topics

The first midterm will cover §1.1 through §1.9 and §2.1 in our textbook, *Linear Algebra and Its Applications*, third edition. The midterm will also cover the assignment problem and the Hungarian method and two-person zero-sum games. Lecture notes on these topics are available on the Schedule page of our course web site. In particular, you will be responsible for the following topics.

§1.1 Systems of Linear Equations

- To translate a system of linear equations into augmented matrix notation and vice versa.
- To determine whether a given equation is linear or not.
- To which row operations can be applied to the augmented matrix of a system of linear equations without changing its solution set.
- To visualize the solution sets of systems of two or three variables.

§1.2 Row Reduction and Echelon Forms

- To determine if a given matrix is in row echelon or reduced row echelon form.
- To determine the pivot positions and pivot columns of a given matrix.
- To use elementary row operations to reduce a given matrix to row echelon or reduced row echelon form.
- To solve a system of linear equations using the row reduction algorithm, describing the solution set parametrically if appropriate.
- To determine the basic and free variables of a system of linear equations.
- To determine if a system of linear equations has no solutions, exactly one solution, or infinitely many solutions.
- To know and be able to apply Theorem 2.

§1.3 Vector Equations

- To simplify expressions involving sums, differences, and scalar multiples of vectors.
- To visualize vectors and sums of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- To determine whether a given vector is a linear combination of a given set of vectors either algebraically and geometrically.
- To translate a system of linear equations from augmented matrix notation to vector equation notation and vice versa.
- To visualize and describe the span of a set of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

§1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

- To compute the matrix-vector product $A\mathbf{x}$.
- To translate a system of linear equations among augmented matrix notation, vector equation notation, and matrix equation notation.
- To know and be able to apply Theorem 4.
- To simplify expressions involving matrix-vector products.

§1.5 Solution Sets of Linear Equations

- To express solutions to systems of linear equations in parametric vector form.
- To visualize and descibe the solution set of a system of linear equations in two or three dimensions.
- To describe the relationship between the solution set of a non-homogeneous system and the solution set of the corresponding homogeneous system.

§1.6 Applications of Linear Systems

- *The Leontief Exchange Model:* Given an economy with many sectors, the total output of each sector for one time period, and how this output is divided among the other sectors, find equilibrium prices that can be assigned to the total outputs of the various sectors so that the income of each sector exactly balances its expenses.
- *Network Flow:* Given information about the flow into and out of each junction in a network, determine and analyze the general flow patten of the network.

§1.7 Linear Independence

- To determine whether a given set of vectors is linearly independent, using the definition of linear independence or some other appropriate technique.
- To find a linear dependence relation among a set of linearly dependent vectors.
- To know and be able to apply Theorem 7.

§1.8 Introduction to Linear Transformations

- To determine the domain, codomain, and range of a given linear transformation.
- To determine if a given linear transformation is onto or one-to-one.
- To determine algebraically and geometrically the effect of a given linear transformation on given vectors.
- To determine if a given transformation is linear.

§1.9 The Matrix of a Linear Transformation

- To use Theorem 10 to find the standard matrix of a given linear transformation.
- To visualize and describe the action of certain linear transformations, including projections, shears, contractions, dilations, rotations, and reflections.
- To determine if a given linear transformation is onto or one-to-one.
- To analyze the existence and uniqueness of solutions to the matrix equation $A\mathbf{x} = \mathbf{b}$ in terms of the linear transformation whose standard matrix is A.

§2.1 Matrix Operations

- To simplify expressions involving sums, differences, scalar products, matrix products, and transpositions of matrices.
- To explain in terms of linear transformations why matrix multiplication is defined as it is.
- To identify ways in which matrix multiplication is not like numerical multiplication.

The Assignment Problem

- To solve an assignment problem in which resources are to be optimally assigned to tasks in a one-to-one manner by modeling the problem as a matrix.
- To use the Hungarian method to solve an assignment problem.

Two-Person Zero Sum Games

- To model a two-person zero-sum game described in words with a payoff matrix.
- To determine the expected payoff of a two-person zero-sum game given its payoff matrix and particular strategies.
- To determine optimal strategies for a strictly determined two-person zero-sum game and the corresponding game value.

2 Suggested Exercises

From the Textbook

- §1.1 #13, 15, 17, 23, 27
- §1.2 #11, 15, 21, 27, 31
- §1.3 #7, 11, 19, 23, 27
- §1.4 #13, 15, 23, 29, 33
- §1.5 #11, 15, 23, 33, 37
- §1.6 #1, 3, 11, 13
- §1.7 #5, 7, 21, 25, 33, 37, 39
- §1.8 #9, 11, 21, 25, 31, 33
- §1.9 #7, 9, 23, 19, 29, 31
- Chapter 1 Supplementary Exercises #1, 7, 13, 17, 23
- §2.1 #1, 11, 15, 21, 23

Additional Exercises:

1. A coin dealer is to sell four coins through a mail auction. Bids are received for each of the four coins from five bidders with instructions from each bidder that at most one of his bids is to be honored. The bids are given in the following table.

	Coin 1	Coin 2	Coin 3	Coin 4
Bidder 1	$150	$65	$210	$135
Bidder 2	175	75	230	155
Bidder 3	135	85	200	140
Bidder 4	140	70	190	130
Bidder 5	170	50	200	160

   How should the dealer assign the four coins in order to maximize the sum of the resulting bids?

2. Two friends (we'll call them Player R and Player C) play a modified version of the two-person game rock-paper-scissors. Each player chooses one of rock, paper, or scissors. Instead of using the usual rock-paper-scissors rules, they assign points each round according to the following payoff matrix.

$R \setminus C$	Rock	Paper	Scissors
Rock	2	-2	0
Paper	-6	0	-5
Scissors	5	2	3

   (a) If Player R chooses paper 50% of the time and scissors 50% of the time, and if Player C chooses rock 75% of the time and scissors 25% of the time, which player will win more rounds of the game on average? Justify your answer.

   (b) If each player chooses an optimal strategy, which player will win more rounds on average? Justify your answer.

Solutions to Additional Exercises:

1. The only optimal solution is the following: Bidder 1 gets coin 3 ($210), bidder 2 gets coin 1 ($175), bidder 3 gets coin 2 ($85), bidder 5 gets coin 4 ($160), and bidder 4 gets nothing for a total of $630.

2. (a) Since the expected payoff given these strategies is -0.625 and this is a negative number, Player C will win more rounds of the game on average.

   (b) Note that the 2 in the third row and second column is a saddle point. Thus, Player R's optimal strategy is to choose scissors each time and Player C's optimal strategy is to choose paper each time. If each player chooses these optimal strategies, then Player R will win every round.