Math 20
Discrete Dynamical Systems

1. **Eigenvector Decomposition Theorem:** Let A be any $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then, for any vector $\mathbf{x}_0$ in $\mathbb{R}^n$, we have
$$A^k\mathbf{x}_0 = a_1\lambda_1^k\mathbf{v}_1 + \cdots + a_n\lambda_n^k\mathbf{v}_n$$
where the scalars $a_1, \ldots, a_n$ are determined by the equation $\mathbf{x}_0 = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$.

2. **Owl Population Example:** The life cycle of spotted owls divides naturally into three stages: juvenile (up to 1 year old), subadult (1 to 2 years), and adult (over 2 years). Let j_k denote the number of juveniles at year k, s_k the number of subadults at year k, and a_k the number of adults at year k.

   Suppose the following relationships hold among j_k, s_k, and a_k.
$$\begin{aligned} j_{k+1} &= 0.33a_k \\ s_{k+1} &= 0.18j_k \\ a_{k+1} &= 0.71s_k + 0.94a_k \end{aligned}$$
   What can we say about the long-term health of the spotted owl population?

3. Let $\mathbf{x}_k = \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix}$ be the population vector at time k. Let $A = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix}$ be the transition matrix. It follows that $\mathbf{x}_k = A^k\mathbf{x}_0$.

4. We can think of this as a Markov chain, it's just that the transition matrix is not stochastic and so there is no guarantee of an eigenvalue 1 and thus no guarantee of a steady-state vector. However, the Eigenvalue Decomposition Theorem can tell us what's happening.

5. It turns out that A has the eigenvalues $\lambda_1 \approx 0.984$, $\lambda_2 \approx -0.022 + 0.206i$, and $\lambda_3 \approx -0.022 - 0.206i$, where $i = \sqrt{-1}$.

6. Whoa! Complex numbers! You can find out about complex numbers in Appendix B and you can find out about complex eigenvalues in §5.5. However, here's what we need to know about them.

7. **Definition:** If $z = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$, the *magnitude* of z is defined to be
$$|z| = \sqrt{a^2 + b^2}.$$

8. **Lemma:** If $z = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$, and $|z| < 1$, then
$$\lim_{k\to\infty} z^k = 0.$$

9. Now let's apply the Eigenvector Decomposition Theorem. Since we have three distinct eigenvectors, we can find three linearly independent eigenvectors for A, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, corresponding to λ_1, λ_2, and λ_3. (See Theorem 2 on page 307.) We don't even need to know what they are right now.

10. The EDT gives us
$$\mathbf{x}_k = A^k\mathbf{x}_0 \approx a_1(0.984)^k\mathbf{v}_1 + a_2\lambda_2^k\mathbf{v}_2 + a_3\lambda_3^k\mathbf{v}_3$$
    for some scalars $a_1, \ldots, a_n$. Note that $|\lambda_1| < 1$, $|\lambda_2| < 1$, and $|\lambda_3| < 1$. This means that
$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{0}.$$
    The owl population will become extinct!

11. We can be more precise. It turns out that $|\lambda_1| > |\lambda_2|$ and $|\lambda_1| > |\lambda_3|$, and so the second and third terms in the linear combination go to 0 faster than the first term does. This means that when k is large,

$$\mathbf{x}_k \approx a_1(0.984)^k \mathbf{v}_1.$$

This tells us that the owl population can be expected to decline by $1 - 0.984 = 0.016 = 1.6\%$ each year.

12. It also tells us that when k is large, the population vector $\mathbf{x}_k$ is proportional to the eigenvector $\mathbf{v}_1$ corresponding to λ_1. It turns out that

$$\mathbf{v}_1 \approx \begin{bmatrix} 0.240 \\ 0.044 \\ 0.716 \end{bmatrix}.$$

This tells us that when k is large, we can expect about 24% of the owls in the population to be juveniles, about 4% to be sub-adults, and about 72% to be adults.