Math 20
§1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Solution 1: Given the row-vector rule for computing $A\mathbf{x}$ found on page 45, we have that

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} 11 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (1)(11) + (4)(2) + (2)(1) \\ (2)(11) + (-1)(2) + (-5)(1) \end{bmatrix} = \begin{bmatrix} 21 \\ 15 \end{bmatrix}.$$

Thus choice 1. is true.

Using the definition of the product $A\mathbf{x}$, we have that

$$\begin{bmatrix} 21 \\ 15 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} 11 \\ 2 \\ 1 \end{bmatrix} = 11 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 21 \\ 15 \end{bmatrix}.$$

Thus choice 3. is true.

The easiest way to show that choice 2. is not true is to consider the first row of the equation given in 2. and note that

$$(b_1)(1) + (b_2)(2) = (21)(1) + (15)(2) \neq 11.$$

Solution 2: As noted in the previous question,

$$\begin{bmatrix} 21 \\ 15 \end{bmatrix} = 11 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 21 \\ 15 \end{bmatrix}.$$

Thus choice 2. is true.

Since the the subset of $\mathbb{R}^2$ spanned by the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ consists of all possible linear combinations of those vectors, and the vector $\begin{bmatrix} 21 \\ 15 \end{bmatrix}$ is a linear combination of those vectors, then choice 3. is true.

As for choice 1., even though the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent for $\mathbf{b} = \begin{bmatrix} 21 \\ 15 \end{bmatrix}$, that does not imply that it is consistent for *any* choice of $\mathbf{b}$. Further analysis would be needed to verify this fact.

Solution 3: Since the system of linear equations corresponding to the augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ is consistent, it follows that there is a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that satisfies the matrix equation $A\mathbf{x} = \mathbf{b}$. Thus,

$$\mathbf{b} = A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$$

and so $\mathbf{b}$ is a linear combination of the columns of A. Thus choice (2) is true.

As for choice (1), Theorem 4 on page 43 tells us that this is true only if the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent *for any* vector $\mathbf{b}$. We are only given that is it consistent for a particular $\mathbf{b}$.

As for choice (3), Theorem 4 on page 43 tells us that this is true only if choice (1) is true.

As for choice (4), this statement only makes sense if A has 3 rows.

Solution 4: The matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the echelon form of the augmented matrix

$$\begin{bmatrix} 1 & 5 & b_1 \\ -2 & -13 & b_2 \\ 3 & -3 & b_3 \end{bmatrix}$$

does not contain a row of the form $\begin{bmatrix} 0 & 0 & c \end{bmatrix}$ where c is any nonzero number. Using the row reduction algorithm, we have

$$\begin{bmatrix} 1 & 5 & b_1 \\ -2 & -13 & b_2 \\ 3 & -3 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & b_1 \\ 0 & -3 & 2b_1 + b_2 \\ 0 & -18 & -3b_1 + b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & b_1 \\ 0 & -3 & 2b_1 + b_2 \\ 0 & 0 & -15b_1 - 6b_2 + b_3 \end{bmatrix}.$$

Thus, the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if

$$-15b_1 - 6b_2 + b_3 = 0.$$

Thus, choice (1) is true.

As for choice (2), the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A. Note that the columns of A are members of $\mathbf{R}^3$ and are not parallel vectors. Thus their span is a plane in $\mathbf{R}^3$. Thus, choice (2) is true.

As verified for choice (1), the echelon form of A does indeed have a row of zeros. Thus choice (3) is true.

As verified for choice (1), the elements of $\mathbf{b}$ must satisfy is certain linear equation. Since not all vectors in $\mathbb{R}^3$ satisfy this equation, choice (4) is not true.