Math 20
§1.5 Solution Sets of Linear Systems

Example 1: Consider the following system of linear equations.

$$\begin{array}{rcrcl} 3x_1 & + & x_2 & = & 7 \\ 2x_1 & + & 4x_2 & = & 8 \end{array}$$

(a) Express this system in vector equation form. What vector problem are we solving here?

(b) Describe this problem geometrically. Does it seem reasonable that there should be a solution? A unique solution?

(c) Suppose that instead of the vector $\begin{bmatrix} 7 \\ 8 \end{bmatrix}$ on the right-hand side, we had some unknown vector $\mathbf{b}$. Would this system be consistent for any vector $\mathbf{b}$? What does this say about the span of the vectors $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$?

(d) Note that the augmented matrix for this system reduces to the following.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Express the solution set of this system in standard and parametric vector form. Describe this solution set geometrically.

Example 2: Consider the following system of linear equations.

$$\begin{array}{rcrcrcl} 2x_1 & + & 5x_2 & + & 3x_3 & = & 10 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 6 \end{array}$$

(a) Express this system in vector equation form. What vector problem are we solving here?

(b) Describe this problem geometrically. Does it seem reasonable that there should be a solution? A unique solution?

(c) Suppose that instead of the vector $\begin{bmatrix} 10 \\ 6 \end{bmatrix}$ on the right-hand side, we had some unknown vector $\mathbf{b}$. Would this system be consistent for any vector $\mathbf{b}$? What does this say about the span of the vectors $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$?

(d) Note that the augmented matrix for this system reduces to the following.

$$\begin{bmatrix} 1 & 0 & -\frac{13}{8} & \frac{5}{2} \\ 0 & 1 & \frac{5}{4} & 1 \end{bmatrix}$$

Express the solution set of this system in standard and parametric vector form. Describe this solution set geometrically.

Example 3: Consider the following system of linear equations.

$$\begin{array}{rcrcl} 3x_1 & + & x_2 & = & 10 \\ 4x_1 & + & 2x_2 & = & 1 \\ x_1 & + & 6x_2 & = & 5 \end{array}$$

(a) Express this system in vector equation form. What vector problem are we solving here?

(b) Describe this problem geometrically. Does it seem reasonable that there should be a solution?

(c) Suppose that instead of the vector $\begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix}$ on the right-hand side, we had some unknown vector $\mathbf{b}$. What property would $\mathbf{b}$ have to have in order for the system to be consistent?

(d) Note that the augmented matrix for this system reduces to the following.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What does this say about the solution set of this system?

Example 4: Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & + & 10x_3 & = & 35 \\ 4x_1 & + & 2x_2 & + & x_3 & = & 11 \\ x_1 & + & 6x_2 & + & 5x_3 & = & 28 \end{array}$$

(a) Express this system in vector equation form. What vector problem are we solving here?

(b) Describe this problem geometrically. Does it seem reasonable that there should be a solution? A unique solution?

(c) Suppose that instead of the vector $\begin{bmatrix} 35 \\ 11 \\ 28 \end{bmatrix}$ on the right-hand side, we had some unknown vector $\mathbf{b}$. Would this system be consistent for any vector $\mathbf{b}$? What does this say about the span of the vectors $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix}$? What relationship is there between this example and the previous one?

(d) Note that the augmented matrix for this system reduces to the following.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Express the solution set of this system in standard and parametric vector form. Describe this solution set geometrically.

Example 5: Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & + & 2x_3 & = & 15 \\ 4x_1 & + & 2x_2 & + & 2x_3 & = & 22 \\ x_1 & + & 6x_2 & - & 5x_3 & = & 22 \end{array}$$

(a) Express this system in vector equation form. What vector problem are we solving here?

(b) Describe this problem geometrically. Does it seem reasonable that there should be a solution? A unique solution?

(c) Note that the augmented matrix for this system reduces to the following.

$$\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Express the solution set of this system in standard and parametric vector form. Describe this solution set geometrically.

(d) Suppose that instead of the vector $\begin{bmatrix} 15 \\ 22 \\ 22 \end{bmatrix}$ on the right-hand side, we had some unknown vector $\mathbf{b}$. Would this system be consistent for any vector $\mathbf{b}$? What does this say about the span of the vectors $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ -5 \end{bmatrix}$?

Example 6: Consider the following system of linear equations.

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & 2x_3 & - & 7x_4 & = & 0 \\ 3x_1 & + & 6x_2 & - & 4x_3 & - & 15x_4 & = & 0 \\ 2x_1 & + & 4x_2 & - & 3x_3 & - & 11x_4 & = & 0 \end{array}$$

(a) What type of linear system is this? Can you immediately find one solution to the system?

(b) Express this system in vector equation form. What vector problem are we solving here?

(c) Describe this problem geometrically. Does it seem reasonable that there should be a solution? A unique solution?

(d) Note that the augmented matrix for this system reduces to the following.

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Express the solution set of this system in standard and parametric vector form. Describe this solution set geometrically.

Example 7: Consider the following system of linear equations.

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & 2x_3 & - & 7x_4 & = & 19 \\ 3x_1 & + & 6x_2 & - & 4x_3 & - & 15x_4 & = & 41 \\ 2x_1 & + & 4x_2 & - & 3x_3 & - & 11x_4 & = & 30 \end{array}$$

(a) Express this system in vector equation form. What vector problem are we solving here?

(b) Describe this problem geometrically. Does it seem reasonable that there should be a solution? A unique solution?

(c) Note that the augmented matrix for this system reduces to the following.

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Express the solution set of this system in standard and parametric vector form. Describe this solution set geometrically.

(d) What relationship is there between this example and the previous one? Does it makes sense that most of the reduced augmented matrices in both examples are identical? What geometric relationship is there between the solution sets of these two examples?