Math 20
§1.6 Applications of Linear Systems

Solution Part a:

$$E = \begin{bmatrix} \frac{7}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{5}{16} & \frac{1}{6} & \frac{5}{16} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

The sum of the entries in each column of the matrix is 1. This is because a sector's column shows where that sector's output goes.

Solution Part b: Note that, for instance, we want the farmers' expenses to equal the farmers income. The total value of all food is p_1 dollars and the farmers buy $\frac{7}{16}$ of all food, so they spend $\frac{7}{16}p_1$ dollars on food. Likewise they spend $\frac{1}{2}p_2$ dollars on shelter and $\frac{3}{16}p_3$ dollars on clothing. Thus the farmers total expenses are equal to

$$\tfrac{7}{16}p_1 + \tfrac{1}{2}p_2 + \tfrac{3}{16}p_3.$$

Since we know that the farmers income equals p_1 dollars, we have that

$$\tfrac{7}{16}p_1 + \tfrac{1}{2}p_2 + \tfrac{3}{16}p_3 = p_1.$$

Likewise,

$$\tfrac{5}{16}p_1 + \tfrac{1}{6}p_2 + \tfrac{5}{16}p_3 = p_2$$

and

$$\tfrac{1}{4}p_1 + \tfrac{1}{3}p_2 + \tfrac{1}{3}p_3 = p_3.$$

Collecting like terms we get the following homogeneous system.

$$\begin{array}{rcrcrcl} -\frac{9}{16}p_1 & + & \frac{1}{2}p_2 & + & \frac{3}{16}p_3 & = & 0 \\ \frac{5}{16}p_1 & - & \frac{5}{6}p_2 & + & \frac{5}{16}p_3 & = & 0 \\ \frac{1}{4}p_1 & + & \frac{1}{3}p_2 & - & \frac{1}{2}p_3 & = & 0 \end{array}$$

Solution Part c: Note that the augmented matrix for this system reduces to the following.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, we have the solution set

$$\begin{aligned} p_1 &= p_3 \\ p_2 &= \tfrac{3}{4}p_3. \end{aligned}$$

As long as these two relationships hold among p_1, p_2, and p_3, then the economy is in equilibrium–each producer is neither gaining nor losing money. This helps with the economy's long-term stability!

If we want the total value of all products to be 2200 dollars, then

$$\begin{aligned} 2200 &= p_1 + p_2 + p_3 \\ &= p_3 + \tfrac{3}{4}p_3 + p_3 \\ &= \tfrac{11}{4}p_3 \end{aligned}$$

and so $p_3 = \frac{4}{11} \cdot 2200 = 800$. It follows that $p_1 = 800$ and $p_2 = \frac{3}{4} \cdot 800 = 600$. These prices would result in equilibrium with a total value of all products of 2200 dollars.