Math 20
§1.7 Linear Independence

Solution 1: *3.* Note that set 1 contains the zero vector, and so by Theorem 1.7.9, the set is linearly dependent. Note that set 2 contains more vectors (four) than there are entries in each vector (three), and so by Theorem 1.7.8, the set is linearly dependent. Note that in set 4, the third vector $\begin{bmatrix} 6 \\ -12 \\ -21 \end{bmatrix}$ is equal to -3 times the second vector $\begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix}$. Since one vector in the set is a linear combination of the prior vectors in the set (in this case, one vector is a scalar multiple of a prior vector), it follows from Theorem 1.7.7 that the set is linearly independent.

This leaves only set 3. To confirm that this is a linear independent set, we must show that the vector equation

$$x_1 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} = \mathbf{0}$$

has only the trivial solution. This vector equation is equivalent to the system of linear equations with augmented matrix

$$\begin{bmatrix} 2 & -2 & 2 & 0 \\ 1 & 4 & -2 & 0 \\ -1 & 7 & 6 & 0 \end{bmatrix}$$

which is row-equivalent to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Since this matrix has a pivot position in all but its rightmost columns, the corresponding system of linear equations has no free variables. Thus, the system can only have a unique solution–the trivial solution $x_1 = x_2 = x_3 = 0$. Thus set 3 is a linearly independent set.

Solution 2: *False.* Consider $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. This is a linear dependent set since

$$0\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}.$$

However, it is clear that $\mathbf{v}_1$ is not a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

Solution 3: *True.* Since the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent, there exist weights c_1, c_2, c_3, and c_4, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}.$$

If c_1 were equal to zero, then at least one of c_2, c_3, and c_4 would have to be nonzero, and so it would follow that

$$c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

for some weights c_2, c_3, and c_4, not all zero. But $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set, so this cannot occur. Thus c_1 cannot equal zero.

Since $c_1 \neq 0$, we can solve the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

for $\mathbf{v}_1$ as follows.

$$\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2 - \frac{c_3}{c_1}\mathbf{v}_3 - \frac{c_4}{c_1}\mathbf{v}_4$$

Thus one can write $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

Solution 4: *2.* From the previous problem, we know that $\mathbf{v}_1$ can be written as a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. Thus,

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$$

for some numbers c_2, c_3, and c_4, not all zero.

Suppose that $\mathbf{u}$ is a vector in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Then

$$\mathbf{u} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + d_4\mathbf{v}_4$$

for some numbers d_1, d_2, d_3, and d_4, not all zero. It follows that

$$\begin{aligned}\mathbf{u} &= d_1\left(c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4\right) + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + d_4\mathbf{v}_4 \\ &= (d_1c_2 + d_2)\mathbf{v}_2 + (d_1c_3 + d_3)\mathbf{v}_3 + (d_1c_4 + d_4)\mathbf{v}_4\end{aligned}$$

and thus $\mathbf{u}$ is also a vector in $\text{Span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Since this holds for any $\mathbf{u}$ in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, we have shown that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a subset of $\text{Span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Since $\text{Span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is clearly a subset of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, it follows that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \text{Span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Note that choice 3 *might* be true. Suppose that $\mathbf{v}_1$ were a scalar multiple of $\mathbf{v}_2$. Then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \text{Span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ since any linear combination involving $\mathbf{v}_2$ could be expressed as a linear combination using $\mathbf{v}_1$ instead, and vice versa.

However, choice 3 *might not* be true. If $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_3$ and $\mathbf{v}_4$ but not $\mathbf{v}_2$, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\} = \text{Span}\{\mathbf{v}_3, \mathbf{v}_4\}$. Why does this not equal $\text{Span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$? Since the set of vectors $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent, it follows that $\mathbf{v}_2$ cannot be written as a linear combination of $\mathbf{v}_3$ and $\mathbf{v}_4$. Hence $\mathbf{v}_2$ is a vector in $\text{Span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ but not a vector in $\text{Span}\{\mathbf{v}_3, \mathbf{v}_4\}$.

Solution 5: *True.* By the definition of linear independence, the columns of A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This follows immediately from what we know about A, since $\mathbf{0}$ is a vector in $\mathbb{R}^m$.

Solution 6: *1.* Note that the conditions on $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ imply that these three vectors are linearly independent. Thus matrix A has the property that its first three columns are linearly independent.

Consider the matrix equation $A\mathbf{x} = \mathbf{0}$. In order for the first three columns of A to be linearly independent, there must be only one set of values for x_1, x_2, and x_3 that satisfy this equation. Thus, none of the variables x_1, x_2, and x_3 can be free variables. The only echelon form with this property is number 1.