Math 20
§1.8 and §1.9 Linear Transformations

1 Linear Transformations

1. **Observation:** Suppose the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by the formula $T(\mathbf{x}) = A\mathbf{x}$, where A is an $m \times n$ matrix.
   (a) Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$. Then
   $$T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}).$$
   (b) Suppose $\mathbf{u}$ is a vector in $\mathbb{R}^n$ and c is a scalar. Then
   $$T(c\mathbf{u}) = A(c\mathbf{u}) = cA\mathbf{u} = cT(\mathbf{u}).$$
2. **Definition:** A transformation T is *linear* if
   (a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in the domain of T and
   (b) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in the domain of T and all scalars c.
3. **Theorem:** Every matrix transformation is a linear transformation.
4. **Proof:** See the above Observation.
5. **Example:** Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation that rotates vectors by 60° counterclockwise. Is T a linear transformation?
6. Linear transformation are said to *preserve* the operations of vector addition and scalar multiplication.
7. **Theorem:** If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$ and
   $$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$
   for all vectors $\mathbf{u}$ and $\mathbf{v}$ in the domain of T and all scalars c and d.
8. **Proof:** Let $\mathbf{u}$ be any vector in the domain of T. Then
   $$T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}.$$
   Also,
   $$T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$
9. **Theorem:** If T is a linear transformation, $c_1, c_2, \ldots, c_p$ are scalars, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in the domain of T, then
   $$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p).$$
10. **Proof:** This theorem follows from repeated applications of the previous theorem.
11. **Question:** Why is a linear transformation called "linear"?

2 The Standard Matrix of a Linear Transformation

1. **Example Continued:** Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation that rotates vectors 60° counter-clockwise about the origin. Find a 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

2. **Theorem 10:** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

   $$T(\mathbf{x}) = A\mathbf{x}$$

   for all $\mathbf{x}$ in $\mathbb{R}^n$. We call A the *standard matrix for the linear transformation* T.

   In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the vector of length n with a 1 in the jth entry and zeroes everywhere else. Thus

   $$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}.$$

3. This theorem implies that every linear transformation is also a matrix transformation. Thus, we may use the terms linear transformation and matrix transformation interchangeably.

4. **Proof:** This theorem is a proved in a manner similar to how we solved the above example.

5. **Question:** Why is a linear transformation called "linear"?

3 Existence and Uniqueness Questions

1. **Theorem 11:** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

2. **Proof:** First suppose that T is one-to-one. Then the transformation T maps at most one input vector in $\mathbb{R}^n$ to the output vector $\mathbf{0}$. Thus the equation $T(\mathbf{x}) = \mathbf{0}$ has at most one solution. Since we know it has the trivial solution, it has only the trivial solution.

   Now suppose that the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. We want to show that the transformation T is one-to-one. Suppose that T maps both the vectors $\mathbf{u}$ and $\mathbf{v}$ to the same output vector $\mathbf{b}$. Then

   $$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

   Since we are supposing that the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution, it follows that $\mathbf{u} - \mathbf{v} = \mathbf{0}$. Thus $\mathbf{u} = \mathbf{v}$. So if T maps two vectors to the same output, those two vectors are really the same vector. Thus, T is one-to-one.

3. **Theorem 12:** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation with standard matrix A. Then

   (a) T is an onto transformation if and only if the columns of A span $\mathbb{R}^m$ and

   (b) T is a one-to-one transformation if and only if the columns of A are linearly independent.

4. **Proof:**

   (a) Note that the following statements are equivalent.
   - The columns of A span $\mathbb{R}^m$.
   - The equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.
   - The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
   - The equation $T(\mathbf{x}) = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
   - The linear transformation T is onto.

   (b) Note that the following statements are equivalent.
   - The columns of A are linearly independent.
   - The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
   - The equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.
   - The linear transformation T is one-to-one.