**Math 20**
**§2.8 Subspaces of $\mathbb{R}^n$**

# 1 Column and Null Spaces

1. Note that set (a) in Question 3 are sets we have seen before. The set of all linear combinations of the columns of a matrix $A$ is the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. We call such set a set the column space of $A$.
2. **Definition:** The *column space* of a matrix $A$ is the set of all linear combinations of the columns of $A$. We denote this set by Col $A$.
3. Our reasoning that set (a) is a subspace of $\mathbb{R}^3$ extends to any column space.
4. **Theorem:** The column space of a $m \times n$ matrix is a subspace of $\mathbb{R}^m$.
5. Our reasoning that set (a) is a subspace of $R^3$ also extends to any set of the form Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
6. **Theorem:** For $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $R^n$, the set Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of $R^n$. We call this set the *subspace spanned* (or *generated*) by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.
7. Note we have also seen sets like set (b) in Question 3, the set of all solutions to a homogeneous matrix equation. We can view this set as the set of all vectors in the domain of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ that are mapped to the zero vector. We all such a set the null space of $A$.
8. **Definition:** The *null space* of a matrix $A$ is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. We denote this set by Nul $A$.
9. Our reasoning that set (b) is a subspace of $\mathbb{R}^3$ extends to any null space.
10. **Theorem 12:** The null space of a $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

# 2 Bases

1. **Definition:** A *basis* for a subspace $H$ of $R^n$ is a linearly independent set in $H$ that spans $H$.
2. See Question 4 and Solution 4 at this point.
3. **Problem:** Find a basis for the null space of the matrix $A = \begin{bmatrix} 1 & 8 & -2 \\ 2 & -8 & 4 \\ 4 & 8 & 0 \end{bmatrix}$.
4. In general, writing the solution set of $A\mathbf{x} = \mathbf{0}$ in parametric vector form identifies a basis for Nul $A$. See Example 5 on page 171 for a good example.
5. **Problem:** Find a basis for the column space of the matrix $A = \begin{bmatrix} 1 & 8 & -2 \\ 2 & -8 & 4 \\ 4 & 8 & 0 \end{bmatrix}$.
6. Note that when $A$ is row reduced to echelon form $B$, the columns are drastically changed, but the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same set of solutions. That is, the columns of $A$ have *exactly the same linear dependence relationships* as the columns of $B$. This gives us the following theorem.
7. **Theorem 13:** The pivot columns of a matrix $A$ form a basis for the column space of $A$.