Math 20
§2.8 Subspaces of $\mathbb{R}^n$

Solution 1: Note that the set (a) consists of all scalar multiples of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Adding two scalar multiples of $\mathbf{v}$ gives another scalar multiple of $\mathbf{v}$, so the set is closed under addition. Multiplying a scalar multiple of $\mathbf{v}$ by another scalar gives a scalar multiple of $\mathbf{v}$, so the set is closed under scalar multiplication. Thus set (a) is a subspace of $\mathbb{R}^2$.

On the other hand, the set (b) contains the vectors $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$, but not the sum of these vectors $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, so the set is *not* closed under addition. Thus the set (b) is not a a subspace of $\mathbb{R}^2$.

As for sets (c) and (d), each consists of all linear combinations of two vectors $\mathbf{u}$ and $\mathbf{v}$. Adding two linear combinations of $\mathbf{u}$ and $\mathbf{v}$ gives another linear combination of $\mathbf{u}$ and $\mathbf{v}$, so the sets are closed under addition. Multiplying a linear combination of $\mathbf{u}$ and $\mathbf{v}$ by a scalar gives another linear combination of $\mathbf{u}$ and $\mathbf{v}$, so the sets are closed under scalar multiplication. Thus the sets (c) and (d) are subspace of $\mathbb{R}^2$.

Solution 2: Note that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in set (a), then each is the zero vector, so their sum is the zero vector and hence in set (a). Thus the set is closed under addition. Likewise, if $\mathbf{u}$ is a vector in set (a) and c is a scalar, then $c\mathbf{u}$ is also the zero vector and hence in set (a). Thus set (a) is a subspace of $\mathbb{R}^2$. In fact, the set consisting of just the zero vector is always a subspace of the space in which it lives.

As for set (b), it consists of all vectors in the first quadrant of $\mathbb{R}^2$ along with the vectors lying along the axes bordering the first quadrant. Any negative scalar multiple of one of these vectors lies outside the first quadrant, and so the set is not closed under scalar multiplication. Thus set (b) is not a subspace of $\mathbb{R}^2$.

Set (c) consists of all vectors in the first and third quadrants of R^2 along with the vectors lying along the axes. Adding a vector from the first quadrant to a vector in the third quadrant can give a vector in the second or fourth quadrant (consider the vectors $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$), so the set is not closed under addition. Thus set (c) is not a subspace of $\mathbb{R}^2$.

Set (d) consists of all vectors in the first and second quadrants of R^2 along with the vectors lying on the axes bordering these quadrants. Any negative scalar multiple of one of these vectors lies outside the first and second quadrants, and so the set is not closed under scalar multiplication. Thus set (d) is not a subspace of $\mathbb{R}^2$.

Solution 3: Set (a) consists of all linear combinations of three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Adding two linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ gives another linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, so the set is closed under addition. Multiplying a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ by a scalar gives another linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, so the set is closed under scalar multiplication. Thus the set (a) is a subspace of $\mathbb{R}^3$.

How can we describe geometrically the set (a)? Note that the second column is equal to 2 times the first column plus -3 times the third column, so set (a) equals Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} \right\}$. This is a plane that passes through the origin and the points $(1, 2, 4)$ and $(-2, 4, 0)$.

Now consider set (b). Suppose that $\mathbf{u}$ and $\mathbf{v}$ are solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. It follows that

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and so $\mathbf{u} + \mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{0}$. Thus the set is closed under addition. Similarly,

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c\mathbf{0} = \mathbf{0}$$

and so $c\mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{0}$. Thus the set is closed under scalar multiplication and so set (b) is a subspace of $\mathbb{R}^3$.

Set (c) is a little tricky because $\mathbb{R}^2$ consists of all vectors with two entries and $\mathbb{R}^3$ consists of all vectors with three entries. Thus $\mathbb{R}^2$ is not even a *subset* of $\mathbb{R}^3$, much less a subspace of $\mathbb{R}^3$. The set

$$\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x \text{ and } y \text{ are real} \right\}$$

is a subset and a subspace of $\mathbb{R}^3$ that "looks" and "acts" like $\mathbb{R}^2$.

Finally, the sum of two vectors in $\mathbb{R}^3$ is a vector in $\mathbb{R}^3$, so $\mathbb{R}^3$ is closed under addition. A scalar multiple of a vector in $\mathbb{R}^3$ is also a vector in $\mathbb{R}^3$, so $\mathbb{R}^3$ is also closed under scalar multiplication. Thus $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Solution 4: Note that the third vector in set (a) is equal to the sum of the first two. Thus set (a) is linearly dependent, and so it is not a basis for R^3.

Note that set (b) contains four vectors in R^3. Thus set (b) is linearly dependent, and so it is not a basis for R^3.

Note that the matrix whose columns are the vectors in set (c) is in echelon form. We can easily see that it has three pivot positions and so the Invertible Matrix Theorem (Theorem 8) tells us that its columns are linearly independent and span R^3. Hence set (c) is a basis for R^3.

Set (d) contains only two vectors. While they are linearly independent, they do not span all of R^3. Thus set (d) is not a basis for R^3.