Math 20
Two-Person Zero-Sum Games

1 Network Television

1. **Question 1:** Two competing television networks, Fox and CBS, are scheduling one-hour blocks of programs in the same time period. Fox can schedule one of three possible programs and CBS can schedule one of four possible programs. Neither network knows which program the other will schedule. Both networks ask the same outside polling agency to give them an estimate of how all possible pairings of the programs will divide the viewing audience. The agency gives them each the following table, whose (i, j)-th entry is the percentage of the viewing audience that will watch Fox if Fox's program i is paired against CBS's program j.

Fox \ CBS	60 Minutes	Survivor	CSI	Raymond
The O.C.	60	20	30	55
House	50	75	45	60
American Idol	70	45	35	30

Which program should your network schedule in order to maximize its viewing audience?

2. **Question 2:** Now suppose you have a spy at your rival network who reports that instead of airing the same show every week, your rival network might change the show it airs in that time slot from week to week. Each of its shows will air for some percentage of the weeks in the season. Your network has committed to only air the show you picked in Part 1 and has not ordered any episodes of the other shows you were considering. Should you be worried that your rival network has outmaneuvered you? Is your strategy from Part 1 still the best strategy?

3. **Question 3:** Now suppose that both networks thought of the idea of changing shows from week to week. Suppose that p_i represents the probability (expressed as a decimal) that Fox will air its show i in a given week, and q_j represents the probability (expressed as a decimal) that CBS will air its show j in a given week. Since p_i and q_j are probabilities, it follows that

$$0 \leq p_i \leq 1 \qquad \text{and} \qquad 0 \leq q_j \leq 1$$

for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. Also,

$$p_1 + p_2 + p_3 = 1 \qquad \text{and} \qquad q_1 + q_2 + q_3 + q_4 = 1.$$

What can Fox expect as its average rating for the season? Give your answer in terms of p_1, p_2, p_3, q_1, q_2, q_3, and q_4.

2 Two-Person Zero-Sum Games

1. **Definition:** A *two-person zero-sum game* is a two-person game in which

   (a) each player has only a finite number of moves (so that all possible outcomes of play and the corresponding gains of players can be displayed in a matrix), and

   (b) in each play of the game the positive gain of one player is equal to the negative gain (loss) of the other player.

2. **Definition:** Given a two-person zero-sum game with players R and C in which player R has m possible moves and player C has n possible moves, let $a_{i,j}$ be the *payoff* that player C makes to player R if player R makes move i and player C makes move j. The matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

is called the *payoff* matrix of the game.

Also let

$$p_i = \text{ probability that player } R \text{ makes move } i$$

and

$$q_i = \text{ probability that player } C \text{ makes move } i.$$

We call the vector $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}$ the *strategy of player* R and the vector $\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}$ the *strategy of player* C.

3. **Definition:** Given strategies $\mathbf{p}$ and $\mathbf{q}$ and payoff matrix A, the *expected payoff* to player R is defined to be

$$\begin{aligned} E(\mathbf{p}, \mathbf{q}) &= a_{1,1}p_1q_1 + a_{1,2}p_1q_2 + \cdots + a_{1,n}p_1q_n + a_{2,1}p_2q_1 + \cdots + a_{m,n}p_mq_n \\ &= \sum_{i,j=1}^{n} a_{i,j}p_iq_j \\ &= \mathbf{p}^T A\mathbf{q}. \end{aligned}$$

4. **Example:** In the network television problem, the payoff matrix was

$$A = \begin{bmatrix} 60 & 20 & 30 & 55 \\ 50 & 75 & 45 & 60 \\ 70 & 45 & 35 & 30 \end{bmatrix}.$$

Suppose Fox aired its shows with equal probability, while CBS aired *60 Minutes* and *Survivor* twice as often as its other shows.

$$\mathbf{p} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \qquad \text{and} \qquad \mathbf{q} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}$$

The expected payoff to Fox would be

$$E(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T A\mathbf{q} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 60 & 20 & 30 & 55 \\ 50 & 75 & 45 & 60 \\ 70 & 45 & 35 & 30 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} = 49\tfrac{13}{18} \approx 49.72$$

5. **Definition:** If $\mathbf{p}^*$ and $\mathbf{q}^*$ are strategies such that

$$E(\mathbf{p}^*, \mathbf{q}) \geq E(\mathbf{p}^*, \mathbf{q}^*) \geq E(\mathbf{p}, \mathbf{q}^*)$$

for all strategies $\mathbf{p}$ and $\mathbf{q}$, then

(a) $\mathbf{p}^*$ is called an *optimal strategy for player* R,

(b) $\mathbf{q}^*$ is called an *optimal strategy for player* C,

(c) and $E(\mathbf{p}^*, \mathbf{q}^*)$ is called the *value* of the game.

6. **Example:** In the network television problem, we considered the strategies corresponding to Fox only airing *House* and CBS only airing *CSI*.

$$\mathbf{p}^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{q}^* = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

In Question 1, we showed that $E(\mathbf{p}^*, \mathbf{q}^*) = 45$. In Question 2, the Fox students showed that $E(\mathbf{p}^*, \mathbf{q}) \geq 45$ for any CBS strategy $\mathbf{q}$ and the CBS students showed that $E(\mathbf{p}, \mathbf{q}^*) \leq 45$ for any Fox strategy $\mathbf{p}$. Thus,

$$E(\mathbf{p}^*, \mathbf{q}) \geq E(\mathbf{p}^*, \mathbf{q}^*) \geq E(\mathbf{p}, \mathbf{q}^*)$$

for the *House* and *CSI* strategies and so these are optimal strategies for Fox and CBS.

7. **Fundamental Theorem of Zero-Sum Games:** There exist optimal strategies $\mathbf{p}^*$ and $\mathbf{q}^*$ for any two-person zero-sum game.

3 Strictly Determined Games

1. **Definition:** An entry $a_{r,s}$ in a payoff matrix A is called a *saddle point* if

(a) $a_{r,s}$ is the smallest entry in its row and

(b) $a_{r,s}$ is the largest entry in its column.

A game whose payoff matrix has a saddle point is called *strictly determined*.

2. **Example:** In the network television problem, the payoff matrix was

$$A = \begin{bmatrix} 60 & 20 & 30 & 55 \\ 50 & 75 & 45 & 60 \\ 70 & 45 & 35 & 30 \end{bmatrix}.$$

Note that the $2, 3$ entry 45 is the smallest entry in its row and the largest entry in its column. Thus it is a saddle point and this game is strictly determined.

3. **Theorem:** If a payoff matrix has a saddle point $a_{r,s}$, then an optimal strategy for player R is to always make move r and an optimal strategy for player C is to always make move s.

4. **Definition:** Such strategies for which only one move is possible are called *pure strategies*. Strategies for which more than one move is possible are called *mixed strategies*.

5. **Example:** In the network television problem, the payoff matrix was

$$A = \begin{bmatrix} 60 & 20 & 30 & 55 \\ 50 & 75 & 45 & 60 \\ 70 & 45 & 35 & 30 \end{bmatrix}.$$

Since the $2,3$ entry 45 is a saddle point, an optimal strategy for Fox is to always air show 2 (*House*) and an optimal strategy for CBS is to always air show 3 (*CSI*).

4 2×2 Matrix Games

1. **Theorem:** Suppose a two-person zero-sum game as the 2×2 payoff matrix $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$. If the game is not strictly determined, then optimal strategies for players R and C are

$$\mathbf{p}^* = \begin{bmatrix} \dfrac{a_{2,2} - a_{2,1}}{a_{1,1} + a_{2,2} - a_{1,2} - a_{2,1}} \\ \dfrac{a_{1,1} - a_{1,2}}{a_{1,1} + a_{2,2} - a_{1,2} - a_{2,1}} \end{bmatrix}$$

and

$$\mathbf{q}^* = \begin{bmatrix} \dfrac{a_{2,2} - a_{1,2}}{a_{1,1} + a_{2,2} - a_{1,2} - a_{2,1}} \\ \dfrac{a_{1,1} - a_{2,1}}{a_{1,1} + a_{2,2} - a_{1,2} - a_{2,1}} \end{bmatrix}.$$

The value of the game is

$$\frac{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}}{a_{1,1} + a_{2,2} - a_{1,2} - a_{2,1}}.$$

2. **Example:** The federal government desires to inoculate its citizens against a certain flu virus. The virus has two strains, and the proportions in which the two strains occur in the virus population is not known. Two vaccines have been developed. Vaccine 1 is 85% effective against strain 1 and 70% effective against strain 2. Vaccine 2 is 60% effective against strain 1 and 90% effective against strain 2. What inoculation policy should the government adopt?

3. **Solution:** We may consider this a two-person game in which player R (the government) desires to make the payoff (the fraction of citizens resistant to the virus) as large as possible, and player C (the virus) desires to make the payoff as small as possible. The payoff matrix is $\begin{bmatrix} .85 & .70 \\ .60 & .90 \end{bmatrix}$.

   Using the above formulas, we find that the optimal strategy for the government is to inoculate $\frac{2}{3}$ of the citizens with vaccine 1 and $\frac{1}{3}$ with vaccine 2. This will guarantee that about 76.7% of the citizens will be resistant to a virus attack regardless of the distribution of the two strains.