Name: SOLUTIONS

# Math 20 Midterm 1
## March 11, 2005

| Problem Number | Possible Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 12 | |
| 3 | 18 | |
| 4 | 14 | |
| 5 | 14 | |
| 6 | 30 | |
| Total | 100 | |

**Directions—Please Read Carefully!** Read each problem carefully and make sure to answer the specific questions asked. Some questions ask you to justify or explain your answers. You must do so on to receive full credit on these questions. Be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. In general, the more of your work you write down, the more easily I can grant you partial credit if your answer is incorrect. You may use a standard scientific calculator on this exam, but no other calculators or aids are allowed. **Good Luck!**

1. (a) Express the solution set of $x_1 - 2x_2 + 4x_3 = 0$ in parametric vector form.

$$x_1 = 2x_2 - 4x_3$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

(b) Describe geometrically the solution set of $x_1 - 2x_2 + 4x_3 = 0$. *Be as specific as you can.*

The solution set is a plane in $\mathbb{R}^3$, specifically the one passing through the points $(0,0,0)$, $(2,1,0)$, and $(-4,0,1)$.

(c) Express the solution set of $x_1 - 2x_2 + 4x_3 = 3$ in parametric vector form.

$$x_1 = 2x_2 + 4x_3 + 3$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_3 + 3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

(d) Describe geometrically the solution set of $x_1 - 2x_2 + 4x_3 = 3$ and compare it geometrically with the solution set you found in part (b). *Be as specific as you can.*

The solution set is the plane described in part (b) shifted 3 units in the positive $x_1$ direction.

2. (a) Find the value of $c$ that makes the following equation true.

$$B^T A^T - (4AB)^T = c(AB)^T$$

$$B^T A^T - (4AB)^T = (AB)^T - 4(AB)^T$$

$$= -3(AB)^T$$

$$\boxed{c = -3}$$

(b) Compute $B^T A^T - (4AB)^T$ for the matrices $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & -2 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 3 \\ -2 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$.

$$\| \\ -3(AB)^T$$

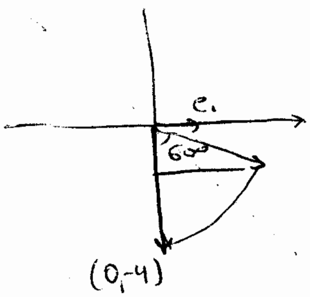
$$AB = \begin{bmatrix} -2 & 0 & 3 \\ -12 & 2 & -4 \\ -4 & 6 & 3 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} -2 & -12 & -4 \\ 0 & 2 & 6 \\ 3 & -4 & 3 \end{bmatrix}$$

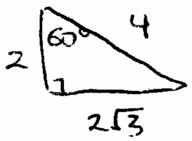
$$-3(AB)^T = \begin{bmatrix} 6 & 36 & 12 \\ 0 & -6 & -18 \\ -9 & 12 & -9 \end{bmatrix}$$

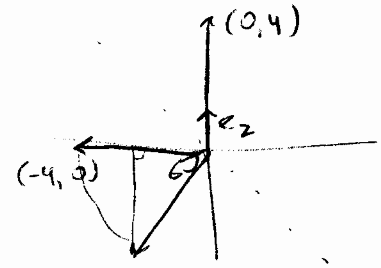
3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that dilates a vector by a factor of 4, then reflects that vector about the line $x_2 = -x_1$, and then rotates that vector $60°$ counterclockwise about the origin.

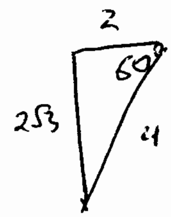
(a) Find the standard matrix for $T$.




$\Rightarrow T(\vec{e}_1) = \begin{bmatrix} 2\sqrt{3} \\ -2 \end{bmatrix}$




$\Rightarrow T(\vec{e}_2) = \begin{bmatrix} -2 \\ -2\sqrt{3} \end{bmatrix}$

$\Rightarrow$ standard matrix

$= \begin{bmatrix} 2\sqrt{3} & -2 \\ -2 & -2\sqrt{3} \end{bmatrix}$

(b) Determine if $T$ is an onto transformation. *Justify your answer.*

A linear transformation is onto if the columns of its standard matrix span its codomain (Thm 12). Since the columns $\begin{bmatrix} 2\sqrt{3} \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -2\sqrt{3} \end{bmatrix}$ are not parallel, they do indeed span $\mathbb{R}^2$.

Thus $T$ is onto.

(c) Determine if $T$ is a one-to-one transformation. *Justify your answer.*

A linear transformation is one-to-one if the columns of its standard matrix are linearly independent. The columns $\begin{bmatrix} 2\sqrt{3} \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -2\sqrt{3} \end{bmatrix}$ are linearly independent because they are not scalar multiples of each other. Thus $T$ is one-to-one.

4. Three homeowners–a carpenter, an electrician, and a plumber–agree to make repairs in their three homes. They agree to work a total of 10 days each according to the following schedule.

| | Work Performed By: | | |
|---|---|---|---|
| | Carpenter | Electrician | Plumber |
| Days of Work in Home of Carpenter | 2 | 1 | 6 |
| Days of Work in Home of Electrician | 4 | 5 | 1 |
| Days of Work in Home of Plumber | 4 | 4 | 3 |

For tax purposes they must report and pay each other a reasonable daily wage, even for the work each does on his or her own home. Their normal daily wages are about \$200, but they agree to adjust their respective daily wages so that each homeowner will come out even, that is, so that the total amount paid out by each is the same as the total amount each receives.

What daily wages should the homeowners choose so that this condition is satisfied and so that each daily wage is close to \$200?

$p_1$ = daily wage of carpenter

$p_2$ = " " " electrician

$p_3$ = " " " plumber

income = expenses

$$\begin{cases} 10p_1 = 2p_1 + p_2 + 6p_3 \\ 10p_2 = 4p_1 + 5p_2 + p_3 \\ 10p_3 = 4p_1 + 4p_2 + 3p_3 \end{cases}$$

$$\Rightarrow \begin{cases} -8p_1 + p_2 + 6p_3 = 0 \\ 4p_1 - 5p_2 + p_3 = 0 \\ 4p_1 + 4p_2 - 7p_3 = 0 \end{cases}$$

$$\begin{bmatrix} -8 & 1 & 6 & 0 \\ 4 & -5 & 1 & 0 \\ 4 & 4 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} -8 & 1 & 6 & 0 \\ 0 & -9 & 8 & 0 \\ 0 & 9 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/8 & -3/4 & 0 \\ 0 & 1 & -8/9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -31/36 & 0 \\ 0 & 1 & -8/9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow p_1 = 31/36\, p_3$

$p_2 = 8/9\, p_3$

$\Rightarrow 31/36\, p_3 + 8/9\, p_3 + p_3 = 600$

$\Rightarrow p_3 \approx \$216$

$p_2 = 8/9 \cdot 216 \approx \$192$

$p_1 \approx 31/36 \cdot 216 \approx \$186$

* You could also set $p_3 = 200$ and compute $p_1$ and $p_2$.

5. A college intends to install air-conditioning in three of its buildings during a one-week spring break. It invites three contractors to submit separate bids for the work involved in each of the three buildings. The bids it receives (in thousand dollar increments) are listed below.

| | Bids: | | |
|---|---|---|---|
| | Building 1 | Building 2 | Building 3 |
| Contractor 1 | 53 | 96 | 37 |
| Contractor 2 | 47 | 87 | 41 |
| Contractor 3 | 60 | 92 | 36 |

Each contractor can install the air-conditioning for only one building during the one-week period, so the college must assign a different contractor to each building. To which building should each contractor be assigned in order to minimize the sum of the corresponding bids? *You must use the Hungarian method to receive full credit for this problem.*

$$\begin{bmatrix} 53 & 96 & 37 \\ 47 & 87 & 41 \\ 60 & 92 & 36 \end{bmatrix} \rightarrow \begin{bmatrix} 16 & 59 & 0 \\ 6 & 46 & 0 \\ 24 & 56 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 10 & 13 & 0 \\ 0 & 0 & 0 \\ 18 & 10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{0} & 3 & 0 \\ 0 & \boxed{0} & 10 \\ 8 & 0 & \boxed{0} \end{bmatrix}$$

optimal solutions :

| Contractor | Building |
|---|---|
| 1 | 3 |
| 2 | 1 |
| 3 | 2 |

or

| Contractor | Building |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |

6. Mark each of the following statements as TRUE or FALSE. (Note that a statement is TRUE if it is always true, and FALSE otherwise.) *Justify your answers.*

(a) If the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is transformed into $\begin{bmatrix} C & \mathbf{d} \end{bmatrix}$ by interchanging the first and second columns of $A$, then the equations $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ have exactly the same solution sets.

FALSE

Interchanging rows of A preserves the solution set, but interchanging columns generally does not. Consider:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{versus} \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

$x_1 = 2$ ← different solutions → $x_1 = 3$

$x_2 = 3$ $\qquad\qquad$ $x_2 = 2$

(b) If the *coefficient matrix* of a system of linear equations has the following echelon form, then the system has infinitely many solutions.

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$$

FALSE

If the augmented column of the system in echelon form has a nonzero entry in the bottom position, then the system is inconsistent.

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & * \end{array}\right] \leftarrow \text{Could be inconsistent.}$$

(c) If $A$ and $B$ are row equivalent $m \times n$ matrices and if the columns of $A$ are linearly independent, then so are the columns of $B$.

TRUE

Since the columns of $A$ are linearly independent, the equation $A\vec{x} = \vec{0}$ has only the trivial solution. Since $A$ and $B$ are row equivalent, the augmented matrices $[A \ \vec{0}]$ and $[B \ \vec{0}]$ are also row equivalent. Thus $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solution sets. Thus $B\vec{x} = \vec{0}$ has only the trivial solution and so the columns of $B$ are linearly independent.

(d) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set, then so is $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}\}$.

FALSE

Consider the linear combination

$$1(\vec{u}-\vec{v}) + 1(\vec{v}-\vec{w}) + 1(\vec{w}-\vec{u})$$
$$= \vec{u} - \vec{v} + \vec{v} - \vec{w} + \vec{w} - \vec{u} = \vec{0}.$$

Since we can write $\vec{0}$ as a nontrivial linear combination of the three vectors, the set is linearly dependent.

(Note that it doesn't matter whether $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.)

(e) If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the linear transformation given by

$$T(\mathbf{x}) = \begin{bmatrix} 4 & 0 \\ -2 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x},$$

then the range of $T$ is the plane in $\mathbb{R}^3$ that passes through the origin and contains the points $(4, -2, 1)$ and $(0, 1, 3)$.

TRUE

The range of $T$ is the span of the columns of its standard matrix and span $\left\{ \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ is indeed the plane described. (Note the columns are not parallel and so their span is a plane. Since $(0,0,0)$, $(4,-2,1)$, & $(0,1,3)$ are all linear combinations of the columns, those points ~~[illegible]~~ lie in the plane.)

(f) If $A$ and $B$ are $n \times n$ matrices, then $(A + B)(A - B) = A^2 - B^2$.

FALSE

$$(A+B)(A-B)$$

$$= A^2 \underbrace{- AB + BA}_{} - B^2$$

$= 0$ only if $AB = BA$, which is not true in general.