

Name: SOLUTIONS

## Math 20 Midterm 2

April 25, 2005

Problem Number	Possible Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

**Directions—Please Read Carefully!** Read each problem carefully and make sure to answer the specific questions asked. Some questions ask you to justify or explain your answers. You must do so on to receive full credit on these questions. Be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. In general, the more of your work you write down, the more easily I can grant you partial credit if your answer is incorrect. You may use a standard scientific calculator on this exam, but no other calculators or aids are allowed. **Good Luck!**

1. Find a basis for the column space of the matrix  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix}$ .

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & -4 \end{bmatrix} \begin{array}{l} \\ R_1 + R_2 \\ -3R_1 + R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ R_2 + R_3 \end{array}$$

$$= B$$

Since  $B$  is in reduced row echelon form, the pivot columns of  $B$  can be seen to form a basis for the column space of  $B$ .

Since elementary row operations preserve the linear dependence relations among the columns of a matrix, those same pivot columns form a basis for the column space of  $A$ .

Thus  $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right\}$  is a

basis for  $\text{col} A$ .

2. Let  $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ . For the following questions, you may use the fact that  $A$  has eigenvalues 1 and 10.

(a) Find a basis for the eigenspace corresponding to the eigenvalue 1.

$$A\vec{x} = \lambda\vec{x} \quad \text{where } \lambda = 1$$

$$\Rightarrow A\vec{x} - \vec{x} = \vec{0}$$

$$= (A - I)\vec{x} = \vec{0}$$

$$A - I = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & 2 & | & 0 \\ 4 & 4 & 2 & | & 0 \\ 2 & 2 & 1 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 + 2x_2 + x_3 &= 0 \\ \Rightarrow x_1 &= -x_2 - \frac{1}{2}x_3 \end{aligned}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

(b) Is  $A$  diagonalizable? Justify your answer.

Since  $\lambda = 10$  is an eigenvalue

of  $A$ , there is an eigenvector  $\vec{v}_1$

of  $A$  corresponding to  $\lambda = 10$ . Then the set

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}, \vec{v}_1 \right\}$  is necessarily linearly independent.  
(By the "Bonus Theorem".)

Thus  $A$  has 3 linearly independent eigenvectors

and hence is diagonalizable.

Basis:  $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$

3. A company produces Web design, software, and networking services. View the company as an open economy described by the accompanying table, where input is in dollars needed for \$1.00 of output. (Note that  $0.125 = \frac{1}{8}$ .)

Purchased From	Input Consumed Per Unit of Output		
	Web Design	Software	Networking
Web Design	\$0.50	\$0.25	\$0.25
Software	\$0.50	\$0.125	\$0.25
Networking	\$0.50	\$0.25	\$0.125

} = C

If the external demand in a given week is the same for each division, which division must produce the most (in terms of dollars) to meet the demand? Justify your answer.

$$\vec{x} = C\vec{x} + \vec{d}$$

↑ production
↑ intermediate demand
↑ final demand

$$\Rightarrow (I - C)\vec{x} = \vec{d} \quad \text{where } \vec{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ (equal demand)}$$

$$\left[ (I - C) \mid \vec{d} \right] = \left[ \begin{array}{ccc|c} 1/2 & -1/4 & -1/4 & 1 \\ -1/2 & 7/8 & -1/4 & 1 \\ -1/2 & -1/4 & 7/8 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 4 & -2 & -2 & 8 \\ -4 & 7 & -2 & 8 \\ -4 & -2 & 7 & 8 \end{array} \right] \begin{array}{l} 8R_1 \\ 8R_2 \\ 8R_3 \end{array}$$

$$\sim \left[ \begin{array}{ccc|c} 4 & -2 & -2 & 8 \\ 0 & 5 & -4 & 16 \\ 0 & -4 & 5 & 16 \end{array} \right] \begin{array}{l} \\ R_1 + R_2 \\ R_1 + R_3 \end{array} \sim \left[ \begin{array}{ccc|c} 4 & -2 & -2 & 8 \\ 0 & 5 & -4 & 16 \\ 0 & 0 & 9/5 & 144/5 \end{array} \right] \begin{array}{l} \\ \\ 4/5 R_2 + R_3 \end{array}$$

$$\sim \left[ \begin{array}{ccc|c} 4 & -2 & -2 & 8 \\ 0 & 5 & -4 & 16 \\ 0 & 0 & 1 & 16 \end{array} \right] \begin{array}{l} \\ \\ 5/4 R_3 \end{array} \sim \left[ \begin{array}{ccc|c} 4 & -2 & 0 & 56 \\ 0 & 5 & 0 & 80 \\ 0 & 0 & 1 & 16 \end{array} \right] \begin{array}{l} 2R_3 + R_2 \\ 4R_3 + R_2 \\ \end{array}$$

$$\sim \left[ \begin{array}{ccc|c} 4 & -2 & 0 & 56 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 16 \end{array} \right] \begin{array}{l} \\ 1/3 R_2 \\ \end{array} \sim \left[ \begin{array}{ccc|c} 4 & 0 & 0 & 72 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 16 \end{array} \right] \begin{array}{l} 2R_2 + R_1 \\ \\ \end{array} \sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 16 \end{array} \right]$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 18 \\ 16 \\ 16 \end{bmatrix} \Rightarrow \boxed{\text{Web Design}}$$

4. A certain experiment produces the data (1, 4.9), (2, 10.8), (3, 27.9), (4, 60.2), and (5, 113).

(a) Describe the model that produces a least-squares fit of these points by a function of the form

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3.$$

Do so by specifying the design vector  $X$ , the observation vector  $y$ , and the parameter vector  $\beta$ .  
Be sure to include the given data in the appropriate places.

$$X = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_5 & x_5^2 & x_5^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix} = \begin{bmatrix} 4.9 \\ 10.8 \\ 27.9 \\ 60.2 \\ 113 \end{bmatrix}$$

$$\vec{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

(b) In terms of the matrices  $X$ ,  $\beta$ , and  $y$ , what equation can be solved in order to find the regression coefficients  $\beta$ ?

$$X^T X \vec{\beta} = X^T \vec{y}$$

5. Mark each of the following statements as TRUE or FALSE. (Note that a statement is TRUE if it is always true, and FALSE otherwise.) *Justify your answers.*

(a) Row operations on a matrix can change its null space.

FALSE

The null space of a matrix  $A$  is the set of solutions to  $A\vec{x} = \vec{0}$ . Since performing row operations on  $[A \mid 0]$  preserves the solutions to  $A\vec{x} = \vec{0}$ , performing row operations on  $A$  preserves its null space.

(b) There is no  $3 \times 3$  matrix whose column space and null space are both lines through the origin.

TRUE

The Rank Theorem states that the dimensions of the column and null spaces of a  $3 \times 3$  matrix must add to 3.

Since a line has dimension 1, a  $3 \times 3$  matrix's null and column spaces can't both be lines because if they were, the dimension sum would only be 2.

(c) The product of two elementary matrices is an elementary matrix.

FALSE

Counter Example:

$E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$  is an elem. matrix because it is obtained from  $I$  by multiplying 1st row by 2 (a single row op).

Similarly,  $F = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$  is an elem. matrix. However

$EF = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  cannot be obtained from  $I$  by a single row op and so  $EF$  is not an elem. matrix.

(d) If  $A$  is a non-invertible  $n \times n$  matrix, then the reduced row echelon form of  $A$  has at least one row of zeros.

TRUE

The Invertible Matrix Theorem states (among other things) that an  $n \times n$  matrix is invertible if and only if it has  $n$  pivot positions. Since  $A$  is not invertible, it must have less than  $n$  pivot positions. Since  $A$  has  $n$  rows, this means that the RRE form of  $A$  has at least one row without a pivot position — this row must be all zeros.