14. $\begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution is (2, –1, 1).

18. Row reduce the augmented matrix corresponding to the given system of three equations:

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

The third equation, $0 = -5$, shows that the system is inconsistent, so the three planes have no point in common.

26. A basic principle of this section is that row operations do not affect the solution set of a linear system. Begin with a simple augmented matrix for which the solution is obviously (–2, 1, 0), and then perform any elementary row operations to produce other augmented matrices. Here are three examples. The fact that they are all row equivalent proves that they all have the solution set (–2, 1, 0).

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & -3 \\ 2 & 0 & 1 & -4 \end{bmatrix}$$

28. Row reduce the augmented matrix for the given system. Scale the first row by $1/a$, which is possible since a is nonzero. Then replace R2 by R2 + $(-c)$R1.

$$\begin{bmatrix} a & b & f \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ 0 & d-c(b/a) & g-c(f/a) \end{bmatrix}$$

The quantity $d - c(b/a)$ must be nonzero, in order for the system to be consistent when the quantity $g - c(f/a)$ is nonzero (which can certainly happen). The condition that $d - c(b/a) \neq 0$ can also be written as $ad - bc \neq 0$, or $ad \neq bc$.

$$6.\ \begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

12. $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -7 & 0 & 6 & 5 \\ 0 & 0 & \textcircled{1} & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:

$$\begin{aligned} \textcircled{x_1} - 7x_2 \quad + 6x_4 &= 5 \\ \textcircled{x_3} - 2x_4 &= -3 \\ 0 &= 0 \end{aligned}$$

Basic variables: x_1 and x_3; free variables: x_2, x_4. General solution: $\begin{cases} x_1 = 5 + 7x_2 - 6x_4 \\ x_2 \text{ is free} \\ x_3 = -3 + 2x_4 \\ x_4 \text{ is free} \end{cases}$

14. $\begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 7 & 0 & 0 & -9 \\ 0 & \textcircled{1} & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:

$$\begin{array}{rcrcrcrcl} x_1 & & & + & 7x_3 & & & = & -9 \\ & & x_2 & - & 6x_3 & - & 3x_4 & = & 2 \\ & & & & & & x_5 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

Basic variables: x_1, x_2, x_5; free variables: x_3, x_4. General solution:

$$\begin{cases} x_1 = -9 - 7x_3 \\ x_2 = 2 + 6x_3 + 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$$

26. Since there are three pivots (one in each row), the augmented matrix must reduce to the fo

$$\begin{bmatrix} \textcircled{1} & 0 & 0 & a \\ 0 & \textcircled{1} & 0 & b \\ 0 & 0 & \textcircled{1} & c \end{bmatrix} \quad \text{and so} \quad \begin{aligned} \textcircled{x_1} &= a \\ \textcircled{x_2} &= b \\ \textcircled{x_3} &= c \end{aligned}$$

No matter what the values of a, b, and c, the solution exists and is unique.

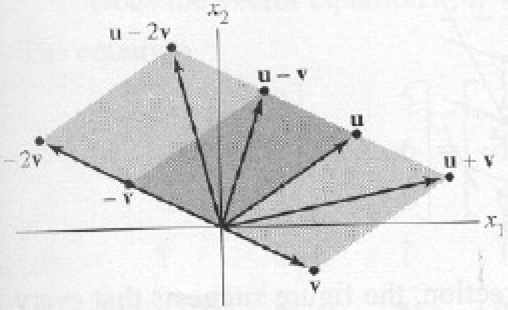
30. Example:

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + 2x_3 &= 5 \end{aligned}$$

32. According to the numerical note in Section 1.2, when $n = 30$ the reduction to echelon form takes about $2(30)^3/3 = 18{,}000$ flops, while further reduction to reduced echelon form needs at most $(30)^2 = 900$ flops. Of the total flops, the "backward phase" is about $900/18900 = .048$ or about 5%.

When $n = 300$, the estimates are $2(300)^3/3 = 18{,}000{,}000$ phase for the reduction to echelon form and $(300)^2 = 90{,}000$ flops for the backward phase. The fraction associated with the backward phase is about $(9\times 10^4)/(18\times 10^6) = .005$, or about .5%.

4.



8. See the figure above. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

w. To reach **w** from the origin, travel –1 units in the **u**-direction (that is, 1 unit in the negative **u**-direction) and travel 2 units in the **v**-direction. Thus, $\mathbf{w} = (-1)\mathbf{u} + 2\mathbf{v}$, or $\mathbf{w} = 2\mathbf{v} - \mathbf{u}$.

x. To reach **x** from the origin, travel 2 units in the **v**-direction and –2 units in the **u**-direction. Thus, $\mathbf{x} = -2\mathbf{u} + 2\mathbf{v}$. Or, use the fact that **x** is –1 units in the **u**-direction from **w**, so that

$$\mathbf{x} = \mathbf{w} - \mathbf{u} = (-\mathbf{u} + 2\mathbf{v}) - \mathbf{u} = -2\mathbf{u} + 2\mathbf{v}$$

y. The vector **y** is 1.5 units from **x** in the **v**-direction, so

$$\mathbf{y} = \mathbf{x} + 1.5\mathbf{v} = (-2\mathbf{u} + 2\mathbf{v}) + 1.5\mathbf{v} = -2\mathbf{u} + 3.5\mathbf{v}$$

z. The map suggests that you can reach **z** if you travel 4 units in the **v**-direction and –3 units in the **u**-direction. So $\mathbf{z} = 4\mathbf{v} - 3\mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$. If you prefer to stay on the paths displayed on the "map," you might travel from the origin to $-2\mathbf{u}$, then 4 units in the **v**-direction, and finally move –1 unit in the **u**-direction. So

$$\mathbf{z} = -2\mathbf{u} + 4\mathbf{v} - \mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$$