Section 1.3

12. The equation

$$x_1\begin{bmatrix}1\\-2\\2\end{bmatrix}+x_2\begin{bmatrix}0\\5\\5\end{bmatrix}+x_3\begin{bmatrix}2\\0\\8\end{bmatrix}=\begin{bmatrix}-5\\11\\-7\end{bmatrix}$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$M=\begin{bmatrix}1&0&2&-5\\-2&5&0&11\\2&5&8&-7\end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M\sim\begin{bmatrix}1&0&2&-5\\0&5&4&1\\0&5&4&3\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&2&-5\\0&\textcircled{5}&4&1\\0&0&0&\textcircled{2}\end{bmatrix}$$

The linear system corresponding to M has *no* solution, so the vector equation (*) has no solution, and therefore **b** is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

14. $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]=\begin{bmatrix}1&-2&-6&11\\0&3&7&-5\\1&-2&5&9\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-2&-6&11\\0&\textcircled{3}&7&-5\\0&0&\textcircled{11}&-2\end{bmatrix}$. The linear system corresponding to this matrix *has* a solution, so **b** is a linear combination of the columns of A.

20. Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane in $\mathbf{R}^3$ through the origin, because the neither vector in this problem is a multiple of the other. Every vector in the set has 0 as its second entry and so lies in the xz-plane in ordinary 3-space. So Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ *is* the xz-plane.

22. Construct any 3×4 matrix in echelon form that corresponds to an inconsistent system. Perform sufficient row operations on the matrix to eliminate all zero entries in the first three columns.

28. **a.** The amount of heat produced when the steam plant burns x_1 tons of anthracite and x_2 tons of bituminous coal is $27.6x_1 + 30.2x_2$ million Btu.

b. The total output produced by x_1 tons of anthracite and x_2 tons of bituminous coal is given by the vector $x_1\begin{bmatrix}27.6\\3100\\250\end{bmatrix}+x_2\begin{bmatrix}30.2\\6400\\360\end{bmatrix}$.

c. **[M]** The appropriate values for x_1 and x_2 satisfy $x_1\begin{bmatrix}27.6\\3100\\250\end{bmatrix}+x_2\begin{bmatrix}30.2\\6400\\360\end{bmatrix}=\begin{bmatrix}162\\23{,}610\\1{,}623\end{bmatrix}$.

To solve, row reduce the augmented matrix:

$$\begin{bmatrix}27.6&30.2&162\\3100&6400&23610\\250&360&1623\end{bmatrix}\sim\begin{bmatrix}1.000&0&3.900\\0&1.000&1.800\\0&0&0\end{bmatrix}$$

The steam plant burned 3.9 tons of anthracite coal and 1.8 tons of bituminous coal.

34. a. For $j = 1,\ldots, n$, $u_j + (-1)u_j = (-1)u_j + u_j = 0$, by properties of **R**. By vector equality,

$$\mathbf{u} + (-1)\mathbf{u} = (-1)\mathbf{u} + \mathbf{u} = \mathbf{0}.$$

b. For scalars c and d, the jth entries of $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are $c(du_j)$ and $(cd)u_j$, respectively. These entries in **R** are equal, so the vectors $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are equal.

Section 1.4

10. The system has the same solution set as the vector equation

$$x_1\begin{bmatrix}8\\5\\1\end{bmatrix}+x_2\begin{bmatrix}-1\\4\\-3\end{bmatrix}=\begin{bmatrix}4\\1\\2\end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix}8&-1\\5&4\\1&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}4\\1\\2\end{bmatrix}$$

12. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix}1&2&1&0\\-3&-1&2&1\\0&5&3&-1\end{bmatrix}\sim\begin{bmatrix}1&2&1&0\\0&5&5&1\\0&5&3&-1\end{bmatrix}\sim\begin{bmatrix}1&2&1&0\\0&5&5&1\\0&0&-2&-2\end{bmatrix}\sim\begin{bmatrix}1&2&1&0\\0&5&5&1\\0&0&1&1\end{bmatrix}$$

$$\sim\begin{bmatrix}1&2&0&-1\\0&5&0&-4\\0&0&1&1\end{bmatrix}\sim\begin{bmatrix}1&2&0&-1\\0&1&0&-4/5\\0&0&1&1\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&0&3/5\\0&\textcircled{1}&0&-4/5\\0&0&\textcircled{1}&1\end{bmatrix}$$

The solution is $\begin{cases}x_1 &= 3/5\\ x_2 &= -4/5\\ x_3 &= 1\end{cases}$. As a vector, the solution is $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}3/5\\-4/5\\1\end{bmatrix}$.

16. Row reduce the augmented matrix $[A\ \ \mathbf{b}]$: $A=\begin{bmatrix}1&-3&-4\\-3&2&6\\5&-1&-8\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$.

$$\begin{bmatrix}1&-3&-4&b_1\\-3&2&6&b_2\\5&-1&-8&b_3\end{bmatrix}\sim\begin{bmatrix}1&-3&-4&b_1\\0&-7&-6&b_2+3b_1\\0&14&12&b_3-5b_1\end{bmatrix}$$

$$\sim\begin{bmatrix}1&-3&-4&b_1\\0&-7&-6&b_2+3b_1\\0&0&0&b_3-5b_1+2(b_2+3b_1)\end{bmatrix}=\begin{bmatrix}\textcircled{1}&-3&-4&b_1\\0&\textcircled{-7}&-6&b_2+3b_1\\0&0&0&b_1+2b_2+b_3\end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $b_1 + 2b_2 + b_3 = 0$. The set of such **b** is a plane through the origin in $\mathbf{R}^3$.

22. Row reduce the matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix}0&0&4\\0&-3&-1\\-2&8&-5\end{bmatrix}\sim\begin{bmatrix}\textcircled{-2}&8&-5\\0&\textcircled{-3}&-1\\0&0&\textcircled{4}\end{bmatrix}$$

The matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ has a pivot in each row, so the columns of the matrix span $\mathbf{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbf{R}^4$.

32. A set of three vectors in cannot span $\mathbf{R}^4$. Reason: the matrix A whose columns are these three vectors has four rows. To have a pivot in each row, A would have to have at least four columns (one for each pivot), which is not the case. Since A does not have a pivot in every row, its columns do not span $\mathbf{R}^4$, by Theorem 4. In general, a set of n vectors in $\mathbf{R}^m$ cannot span $\mathbf{R}^m$ when n is less than m.

34. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the associated system of equations does not have any free variables. If every variable is a basic variable, then each column of A is a pivot column. So the reduced echelon form of A must be $\begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$. Now it is clear that A has a pivot position in each *row*.

By Theorem 4, the columns of A span $\mathbf{R}^3$.

Assignment Worksheet