

$$6. \begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 4 & 0 \\ 0 & \textcircled{1} & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\textcircled{x_1} + 4x_3 = 0$$

$$\textcircled{x_2} - 3x_3 = 0. \text{ The variable } x_3 \text{ is free, } x_1 = -4x_3, \text{ and } x_2 = 3x_3.$$

$$0 = 0$$

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}.$

12.

$$\begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 5 & 0 & 8 & 1 & 0 & 0 \\ 0 & 0 & \textcircled{1} & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\textcircled{x_1} + 5x_2 + 8x_4 + x_5 = 0$$

$$\textcircled{x_3} - 7x_4 + 4x_5 = 0$$

$$\textcircled{x_6} = 0$$

$$0 = 0$$

The basic variables are x_1 , x_3 , and x_6 ; the free variables are x_2 , x_4 , and x_5 . The general solution is $x_1 = -5x_2 - 8x_4 - x_5$, $x_3 = 7x_4 - 4x_5$, and $x_6 = 0$. In parametric vector form, the solution is

$$\begin{aligned}
 \mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5x_2 - 8x_4 - x_5 \\ x_2 \\ 7x_4 - 4x_5 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} -5x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -8x_4 \\ 0 \\ 7x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_5 \\ 0 \\ 4x_5 \\ 0 \\ x_5 \end{bmatrix} \\
 &= x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}
 \end{aligned}$$

16. Row reduce the augmented matrix for the system:

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 4 & -5 \\ 0 & \textcircled{1} & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\textcircled{x_1} + 4x_3 = -5$$

$\textcircled{x_2} - 3x_3 = 3$. Thus $x_1 = -5 - 4x_3$, $x_2 = 3 + 3x_3$, and x_3 is free. In parametric vector form,
 $0 = 0$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 - 4x_3 \\ 3 + 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

The solution set is the line through $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$, parallel to the line that is the solution set of the homogeneous system in Exercise 6.

18. Solve $x_1 - 3x_2 + 5x_3 = 4$ for the basic variable: $x_1 = 4 + 3x_2 - 5x_3$, with x_2 and x_3 free. In vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

The solution of $x_1 - 3x_2 + 5x_3 = 0$ is $x_1 = 3x_2 - 5x_3$, with x_2 and x_3 free. In vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = x_2 \mathbf{u} + x_3 \mathbf{v}$$

The solution set of the homogeneous equation is the plane through the origin in \mathbb{R}^3 spanned by \mathbf{u} and \mathbf{v} . The solution set of the nonhomogeneous equation is parallel to this plane and passes through the

point $\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$.

22. The line through \mathbf{p} and \mathbf{q} is parallel to $\mathbf{q} - \mathbf{p}$. So, given $\mathbf{p} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$, form

$$\mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 - (-6) \\ -4 - 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}, \text{ and write the line as } \mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + t \begin{bmatrix} 6 \\ -7 \end{bmatrix}$$

26. (*Geometric argument using Theorem 6.*) Since $A\mathbf{x} = \mathbf{b}$ is consistent, its solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, by Theorem 6. So the solution set of $A\mathbf{x} = \mathbf{b}$ is a single vector if and only if the solution set of $A\mathbf{x} = \mathbf{0}$ is a single vector, and that happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*Proof using free variables.*) If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of A is a pivot column. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

38. No. If $A\mathbf{x} = \mathbf{y}$ has no solution, then A cannot have a pivot in each row. Since A is 3×3 , it has at most two pivot positions. So the equation $A\mathbf{x} = \mathbf{z}$ for any \mathbf{z} has at most two basic variables and at least one free variable. Thus, the solution set for $A\mathbf{x} = \mathbf{z}$ is either empty or has infinitely many elements.

2. Take some other value for p_S , say 200 million dollars. The other equilibrium prices are then $p_C = 188$ million, $p_E = 170$ million. Any constant nonnegative multiple of these prices is a set of equilibrium prices, because the solution set of the system of equations consists of all multiples of one vector. Changing the unit of measurement to, say, European euros has the same effect as multiplying all equilibrium prices by a constant. The *ratios* of the prices remain the same, no matter what currency is used.

4. a. Fill in the exchange table one column at a time. The entries in each column must sum to 1.

Distribution of Output From:

	Agric.	Energy	Manuf.	Transp.	input	<u>Purchased by:</u>
output	↓	↓	↓	↓		
	.65	.30	.30	.20	→	Agric.
	.10	.10	.15	.10	→	Energy
	.25	.35	.15	.30	→	Manuf.
	0	.25	.40	.40	→	Transp.

- b. Denote the total annual output of the sectors by p_A , p_E , p_M , and p_T , respectively. From the first row of the table, the total input to Agriculture is $.65p_A + .30p_E + .30p_M + .20p_T$. So the equilibrium prices must satisfy

income

expenses

$$p_A = .65p_A + .30p_E + .30p_M + .20p_T$$

From the second, third, and fourth rows of the table, the equilibrium equations are

$$p_E = .10p_A + .10p_E + .15p_M + .10p_T$$

$$p_M = .25p_A + .35p_E + .15p_M + .30p_T$$

$$p_T = .25p_E + .40p_M + .40p_T$$

Move all variables to the left side and combine like terms:

$$.35p_A - .30p_E - .30p_M - .20p_T = 0$$

$$-.10p_A + .90p_E - .15p_M - .10p_T = 0$$

$$-.25p_A - .35p_E + .85p_M - .30p_T = 0$$

$$-.25p_E - .40p_M + .60p_T = 0$$

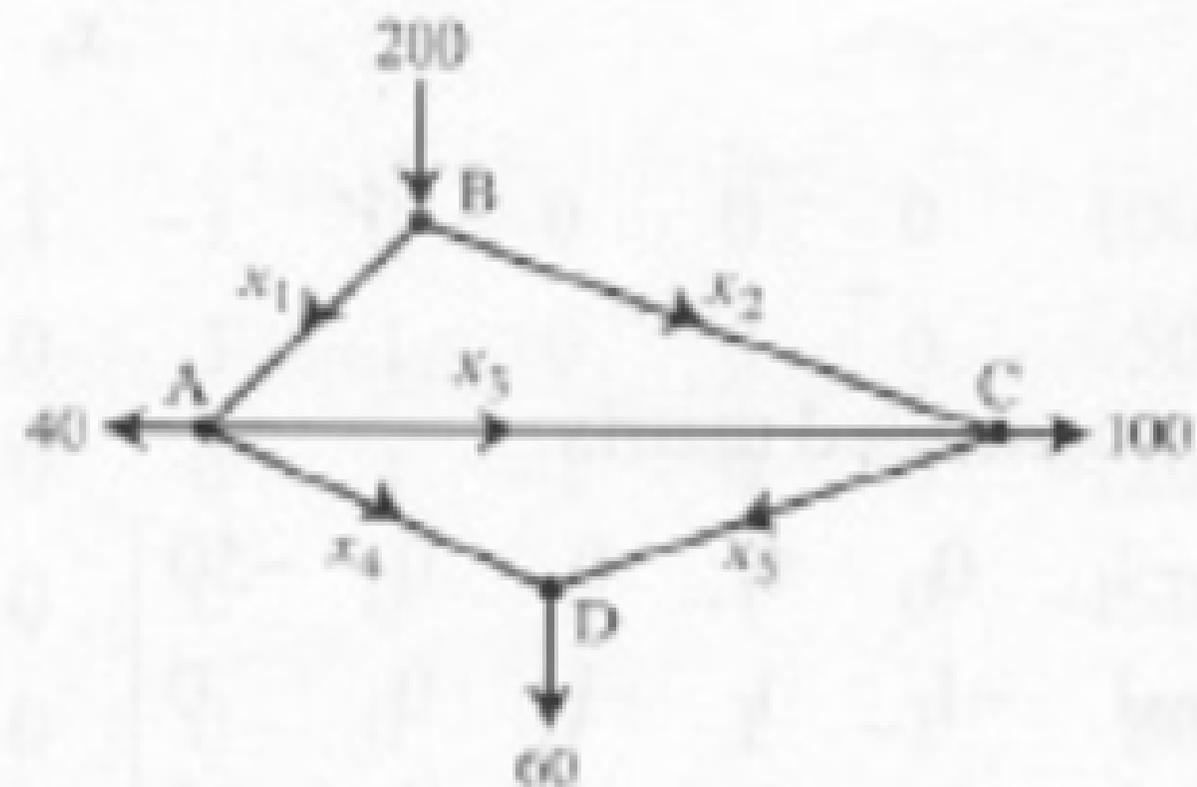
Use gauss, bgauss, and scale operations to reduce the augmented matrix to reduced echelon form

$$\begin{bmatrix} .35 & -.3 & -.3 & -.2 & 0 \\ 0 & .81 & -.24 & -.16 & 0 \\ 0 & 0 & 1.0 & -1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} .35 & -.3 & 0 & -.55 & 0 \\ 0 & .81 & 0 & -.43 & 0 \\ 0 & 0 & 1 & -1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{.35} & 0 & 0 & -.71 & 0 \\ 0 & \textcircled{1} & 0 & -.53 & 0 \\ 0 & 0 & \textcircled{1} & -1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Scale the first row and solve for the basic variables in terms of the free variable p_T , and obtain $p_A = 2.03p_T$, $p_E = .53p_T$, and $p_M = 1.17p_T$. The data probably justifies at most two significant figures, so take $p_T = 100$ and round off the other prices to $p_A = 200$, $p_E = 53$, and $p_M = 120$.

11. Write the equations for each intersection:

Intersection	Flow in	=	Flow out
A	x_1	=	$x_3 + x_4 + 40$
B	200	=	$x_1 + x_2$
C	$x_2 + x_3$	=	$x_5 + 100$
D	$x_4 + x_5$	=	60
Total flow:	200	=	200



Rearrange the equations:

$$\begin{array}{rccccrcr} x_1 & & & - x_3 & - x_4 & & = & 40 \\ x_1 & + & x_2 & & & & = & 200 \\ & & x_2 & + & x_3 & & - & x_5 = & 100 \\ & & & & & x_4 & + & x_5 = & 60 \end{array}$$

Reduce the augmented matrix:

$$\left[\begin{array}{cccccc} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array} \right] \sim \left[\begin{array}{cccccc} \textcircled{1} & 0 & -1 & 0 & 1 & 100 \\ 0 & \textcircled{1} & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

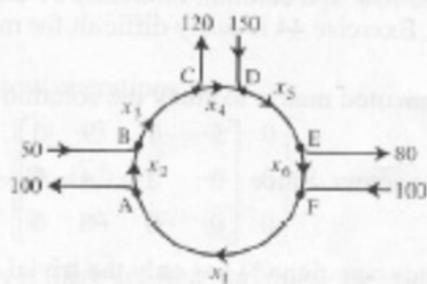
The general solution (written in the style of Section 1.2) is

$$\left\{ \begin{array}{l} x_1 = 100 + x_3 - x_5 \\ x_2 = 100 - x_3 + x_5 \\ x_3 \text{ is free} \\ x_4 = 60 - x_5 \\ x_5 \text{ is free} \end{array} \right. \quad \text{b. When } x_4 = 0, x_5 \text{ must be } 60, \text{ and } \left\{ \begin{array}{l} x_1 = 40 + x_3 \\ x_2 = 160 - x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \\ x_5 = 60 \end{array} \right.$$

c. The minimum value of x_1 is 40 cars/minute, because x_3 cannot be negative.

14. Write the equations for each intersection.

Intersection	Flow in	Flow out
A	x_1	$= x_2 + 100$
B	$x_2 + 50$	$= x_3$
C	x_3	$= x_4 + 120$
D	$x_4 + 150$	$= x_5$
E	x_5	$= x_6 + 80$
F	$x_6 + 100$	$= x_1$



Rearrange the equations:

$$\begin{array}{rcl}
 x_1 - x_2 & & = 100 \\
 x_2 - x_3 & & = -50 \\
 x_3 - x_4 & & = 120 \\
 x_4 - x_5 & & = -150 \\
 x_5 - x_6 & & = 80 \\
 -x_1 & + x_6 & = -100
 \end{array}$$

Reduce the augmented matrix:

$$\left[\begin{array}{cccccc|c}
 1 & -1 & 0 & 0 & 0 & 0 & 100 \\
 0 & 1 & -1 & 0 & 0 & 0 & -50 \\
 0 & 0 & 1 & -1 & 0 & 0 & 120 \\
 0 & 0 & 0 & 1 & -1 & 0 & -150 \\
 0 & 0 & 0 & 0 & 1 & -1 & 80 \\
 -1 & 0 & 0 & 0 & 0 & 1 & -100
 \end{array} \right] \sim \dots \sim \left[\begin{array}{cccccc|c}
 1 & -1 & 0 & 0 & 0 & 0 & 100 \\
 0 & 1 & -1 & 0 & 0 & 0 & -50 \\
 0 & 0 & 1 & -1 & 0 & 0 & 120 \\
 0 & 0 & 0 & 1 & -1 & 0 & -150 \\
 0 & 0 & 0 & 0 & 1 & -1 & 80 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array} \right]$$

$$\dots \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ The general solution is } \begin{cases} x_1 = 100 + x_6 \\ x_2 = x_6 \\ x_3 = 50 + x_6 \\ x_4 = -70 + x_6 \\ x_5 = 80 + x_6 \\ x_6 \text{ is free} \end{cases}$$

Since x_4 cannot be negative, the minimum value of x_6 is 70.

Note: The MATLAB box in the *Study Guide* discusses rational calculations, needed for balanced chemical equations in Exercises 9 and 10. As usual, the appendices cover this material for Mathematica, and the TI and HP graphic calculators.

6. Use the method of Example 2. Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & -3 & 12 & 0 \\ 0 & 4 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 0 \\ 0 & \textcircled{-1} & 4 & 0 \\ 0 & 0 & \textcircled{7} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are no free variables. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so the columns of A are linearly independent.

8. Same situation as with Exercise 7. The (unnecessary) row operations are

$$\begin{bmatrix} 1 & -3 & 3 & -2 & 0 \\ -3 & 7 & -1 & 2 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & -2 & 0 \\ 0 & -2 & 8 & -4 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 3 & -2 & 0 \\ 0 & \textcircled{-2} & 8 & -4 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \end{bmatrix}$$

Again, because there are at most three pivot positions yet there are four variables, the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and the columns of A are linearly dependent.

14. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{0}]$:

$$\begin{bmatrix} -1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ -3 & 8 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -7 & h+3 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -5 & 1 & 0 \\ 0 & \textcircled{2} & 2 & 0 \\ 0 & 0 & h+10 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h + 10 = 0$ (which corresponds to x_3 being a free variable). Thus, the vectors are linearly dependent if and only if $h = -10$.